

# Achievable rate regions for bi-directional relaying

Sang Joon Kim, Natasha Devroye, Patrick Mitran, and Vahid Tarokh

## Abstract

In a bi-directional relay channel, two nodes wish to exchange independent messages over a shared wireless half-duplex channel with the help of a relay. In this paper, we derive achievable rate regions for four new half-duplex protocols and compare these to four existing half-duplex protocols and outer bounds. In time, our protocols consist of either two or three phases. In the two phase protocols, both users simultaneously transmit during the first phase and the relay alone transmits during the second phase, while in the three phase protocol the two users sequentially transmit followed by a transmission from the relay. The relay may forward information in one of four manners; we outline existing Amplify and Forward (AF), Decode and Forward (DF) and Compress and Forward (CF) relaying schemes and introduce the novel Mixed Forward scheme. The latter is a combination of CF in one direction and DF in the other. We derive achievable rate regions for the CF and Mixed relaying schemes for the two and three phase protocols. In the last part of this work we provide a comprehensive treatment of 8 possible half-duplex bi-directional relaying protocols in Gaussian noise, obtaining their respective achievable rate regions, outer bounds, and their relative performance under different SNR and relay geometries.

## Index Terms

bi-directional communication, achievable rate regions, compress and forward, relaying

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## I. INTRODUCTION

Bi-directional relay channels, or wireless channels in which two nodes (a and b)<sup>1</sup> wish to exchange independent messages with the help of a third relay node r, are both of fundamental and practical interest. Such channels may be relevant to ad hoc networks as well as to networks with a centralized controller through which all messages must pass. From an information theoretic perspective, an understanding of these fundamental bi-directional channels would bring us closer to a coherent picture of multi-user information theory. To this end, we study bi-directional relay channels with the goal of determining spectrally efficient achievable rate regions and tight outer bounds to the capacity region.

This two-way channel [4] was first considered in [22] where an achievable rate region and an outer bound for the case in which nodes operate in full-duplex were obtained. Non-orthogonal<sup>2</sup> full-duplex operation requires nodes to transmit and receive on the same antenna and frequency simultaneously. However, it may not be practically feasible to do so since the intensity of the near field of the transmitted signal is much higher than that of the far field of the received signal. In this work, we thus consider half-duplex communication in which a node may either transmit or receive at some time, but not both. Our goal is to determine spectrally efficient (measured in bits per channel use) transmission schemes and outer bounds for the half-duplex bi-directional relay channel and to compare their performance in a number of scenarios. These scenarios highlight the fact that different protocols may be optimal under different channel conditions.

An obvious half-duplex bi-directional relay protocol is the four phase protocol,  $a \rightarrow r$ ,  $r \rightarrow b$ ,  $b \rightarrow r$  and  $r \rightarrow a$ , where the phases are listed chronologically. However, this protocol is spectrally inefficient and does not take full advantage of the broadcast nature of the wireless channel. One way to take advantage of the shared wireless medium would be to combine the second and the fourth phases into a single broadcast transmission by using, for example, network coding [1]. That is, if the relay r can decode the messages  $w_a$  and  $w_b$  from nodes a and b respectively, it is sufficient for the relay r to broadcast  $w_a \oplus w_b$  to both a and b.

In this paper we consider two possible bi-directional relay protocols which differ in their number of phases. Throughout this work, *phases* will denote temporal phases, or durations. The three phase protocol is called the *Time Division Broadcast* (TDBC) protocol, while the two phase protocol is called

<sup>1</sup>We call the nodes a and b *terminal* and *source* nodes interchangeably.

<sup>2</sup>By non-orthogonal we mean that no additional space, time, frequency, or coding dimensions are used to separate sent and receive signals.

the *Multiple Access Broadcast* (MABC) protocol. One of the main conceptual differences between these two protocols is the possibility of *side-information* in the TDBC protocol but not in the MABC protocol. By side-information we mean information obtained from the wireless channel in a particular phase which may be combined with information obtained in different stages to potentially improve decoding or increase transmission rates. The two considered protocols may be described as:

- 1) TDBC protocol: this consists of the three phases  $a \rightarrow r$ ,  $b \rightarrow r$  and  $a \leftarrow r \rightarrow b$ . In this protocol, only a single node is transmitting at any given point in time. Therefore, by the broadcast nature of the wireless channel, the non-transmitting nodes may listen in and obtain “side information” about the transmissions of the other nodes. This may be used for more efficient decoding, i.e. improved rates.
- 2) MABC protocol: this protocol combines the first two phases of the TDBC protocol and consists of the two phases  $a \rightarrow r \leftarrow b$  and  $a \leftarrow r \rightarrow b$ . Due to the half-duplex assumption, during phase 1 both source nodes are transmitting and thus cannot obtain any “side information” regarding the other nodes’ transmission. It may nonetheless be spectrally efficient since it has less phases than the TDBC protocol and may take advantage of the multiple-access channel in phase 1.

We consider restricted protocols in the sense that the receivers must decode their messages at the end of the third phase (TDBC) or second phase (MABC) and collaboration accross multiple successive runs of the protocols are not possible. For each of the MABC and TDBC protocols, the relay may process and forward the received signals differently. These different forwarding schemes are motivated by different *relaying capabilities or assumptions* (about the required complexity or knowledge). Combining the relaying schemes with the temporal protocols, we can obtain various protocols whose rate regions are not in general subsets of one another. The relative benefits and merits of the two protocols and four relaying schemes are summarized in Tables I and II. The four relaying schemes we consider are:

- 1) *Amplify and Forward* (AF): the relay  $r$  constructs its symbol by symbol replication of the received symbol. The AF scheme does not require any computation for relaying, and carries the noise incurred in the first stage(s) forward during the latter relaying stage.
- 2) *Decode and Forward* (DF): the relay decodes both messages from nodes  $a$  and  $b$  before re-encoding them for transmission. The DF scheme requires the full codebooks of both  $a$  and  $b$  and a large amount of computation at the relay  $r$ .
- 3) *Compress and Forward* (CF): the relay does not decode the messages of  $a$  and  $b$ , nor does it simply amplify the received signal, but it performs something in between these two extremes. It *compresses*

the received signal, which it then transmits. To do so, the relay does not require the codebooks of the source nodes, but it does require the channel output distribution  $p(y_r)$  at the relay.

- 4) *Mixed Forward*: the relay decodes and forwards (DF) the data traveling in one direction (from  $a \rightarrow b$ ), while it compresses and forwards (CF) the data traveling in the opposite direction (from  $a \leftarrow b$ ). For the mixed scheme, one of the codebooks and the channel output distribution are needed at the relay.

In the CF scheme the relay searches the compression codebook to find an appropriate codeword. While the search operation is similar to the decoding operation in the DF scheme, the CF scheme may be less complex since the relay can choose a codebook for compression whose search space is smaller than the DF codebook.

Some of these protocols and relaying schemes have been considered in the past. In [12], the DF TDBC protocol is considered. There, network coding in  $\mathbb{Z}_2^k$  is used to encode the message of relay  $r$  from the estimated messages  $\tilde{w}_a$  and  $\tilde{w}_b$ . The works of [18] and [19] consider the MABC protocol, where an amplification and denoising relaying scheme are introduced. In [14] a lattice code is used for the Gaussian channel in the MABC protocol. The capacity region of the broadcast phase in the MABC protocol assuming the relay has both messages  $w_a$  and  $w_b$  is found in [16]. In [23], [15] Slepian-Wolf coding is extended to lossy broadcast channels with side information at the receivers. In [10], achievable rate regions and outer bounds of the MABC protocol and the TDBC protocol with the DF relaying scheme are derived. There, network coding and random binning are the techniques employed to determine achievable rate regions. Uni-directional CF relaying in the full-duplex channel is first introduced in [5]. An achievable region in the CF MABC protocol is derived in [21]. In [8] a comparison between DF and CF schemes in full-duplex channels is performed, while in [20] a comparison of AF and DF schemes with two relays in the MABC protocol is performed.

In this paper, we derive achievable regions for new CF and mixed relaying schemes in both the TDBC and MABC half-duplex protocols. We also obtain outer bounds for the TDBC and MABC protocols based on cut-set bounds. We compare the achievable rate regions of these four novel schemes with the regions and outer bounds derived in [10] as well as a simple AF scheme in Gaussian noise. We thus present a comprehensive overview of the bi-directional relay channel which highlights the relative performance and tradeoffs of the different schemes under different channel conditions and relay processing capabilities. Notably, we find that under some channel conditions the mixed TDBC protocol outperforms the other protocols and similarly, there are channel conditions for which the CF TDBC protocol has the best performance.

TABLE I  
COMPARISON BETWEEN TWO PROTOCOLS

Protocol	Side information	Number of phases	Interference
MABC	not present	2	present
TDBC	present	3	not present

TABLE II  
COMPARISON BETWEEN FOUR RELAYING SCHEMES

Relaying	Complexity	Noise at relay	Relay needs
AF	very low	carried plus noise at rx	nothing
DF	high	perfectly eliminated	full codebooks
CF	low	carried plus distortion	$p(y_r)$
Mixed	moderate	partially carried	one codebook, $p(y_r)$

This paper is structured as follows: in Section II, we introduce our notation, review previously determined achievable rate regions and outer bounds and define the protocols that we will consider. In Section III we derive achievable rate regions for the CF and mixed relaying schemes. In Section IV we obtain explicit expressions for these, and previous rate regions and outer bounds in Gaussian noise. In Section V, we numerically compute these bounds in the Gaussian noise channel and compare the results for different powers and channel conditions.

## II. PRELIMINARIES

In this work, we will determine and compare the rate regions of eight bi-directional relay protocols. We consider the 2 phase Multiple Access and Broadcast (MABC) and the 3 phase Time Division Broadcast (TDBC) protocol versions of Amplify and Forward (AF), Decode and Forward (DF), Compress and Forward (CF) as well as a Mixed scheme which combines Decode and Forward in one direction with Compress and Forward in the other. The AF, DF protocol regions and the CF MABC protocol region have been derived in prior work [10], [19], [21] while the CF and Mixed protocol regions described in Section III are determined here. Also we slightly improve upon the CF MABC protocol region in [21]. We formally define our notation and problem next.

TABLE III  
INPUT AND OUTPUT ALPHABETS

	MABC	TDBC
Phase 1	$\mathcal{X}_a, \mathcal{X}_b, \mathcal{Y}_r, \hat{\mathcal{Y}}_r$	$\mathcal{X}_a, \mathcal{Y}_b, \mathcal{Y}_r, \hat{\mathcal{Y}}_r$ <sup>3</sup>
Phase 2	$\mathcal{X}_r, \mathcal{Y}_a, \mathcal{Y}_b$	$\mathcal{X}_b, \mathcal{Y}_a, \mathcal{Y}_r, \hat{\mathcal{Y}}_r$
Phase 3	N/A	$\mathcal{X}_r, \mathcal{Y}_a, \mathcal{Y}_b$

### A. Notation and Definitions

We consider two terminal nodes a and b, and one relay node r. Terminal node a (resp. b) has its own message that it wishes to send to the opposite terminal node, node b (resp. a). The relay node r may assist in the bi-directional endeavor. This paper will determine achievable rate regions for bi-directional relay protocols over half-duplex, discrete-time memoryless channels. The half-duplex constraint implies that a node cannot simultaneously transmit and receive data.

We first start with a somewhat more general formulation of the problem that simplifies the application of cut-set outer bounds and then apply it the MABC and TDBC protocols considered here. We consider an  $m$  node set, denoted as  $\mathcal{M} := \{1, 2, \dots, m\}$  (where  $:=$  means defined as). We use  $R_{i,j}$  to denote the transmitted data rate of message  $W_{i,j}$  from node  $i \in \mathcal{M}$  to node  $j \in \mathcal{M}$ , i.e.,  $W_{i,j} \in \{0, \dots, \lfloor 2^{nR_{i,j}} \rfloor - 1\} := \mathcal{S}_{i,j}$ . The protocols considered have either  $L = 2$  (MABC) or  $L = 3$  (TDBC) *phases*. We denote by  $\Delta_\ell \geq 0$  the relative time duration of the  $\ell^{th}$  phase, where  $\sum_\ell \Delta_\ell = 1$ . For a given block size  $n$ ,  $\Delta_{\ell,n}$  denotes the duration of the  $\ell^{th}$  phase. Obviously,  $\Delta_{\ell,n} \rightarrow \Delta_\ell$  as  $n \rightarrow \infty$ .

For notational convenience, we define the messages  $W_a := W_{a,b}$ ,  $W_b := W_{b,a}$  and the corresponding rates  $R_a := R_{a,b}$  and  $R_b := R_{b,a}$ . The two distinct messages  $W_a$  and  $W_b$  are taken to be independent and uniformly distributed in the set of  $\{0, \dots, \lfloor 2^{nR_a} \rfloor - 1\} := \mathcal{S}_a$  and  $\{0, \dots, \lfloor 2^{nR_b} \rfloor - 1\} := \mathcal{S}_b$ , respectively.

We use channel input alphabet  $\mathcal{X}_i$  and channel output alphabet  $\mathcal{Y}_i$  for node  $i$ . We will be constructing Compress and Forward schemes in which received signals are compressed or quantized before being re-transmitted. We let  $\hat{Y}_i$  denote the compressed representation of the received signal at node  $i$ , which lies in the corresponding compression alphabet  $\hat{\mathcal{Y}}_i$  for node  $i$ .  $\hat{\mathcal{Y}}_i$  is not necessarily equal to  $\mathcal{Y}_i$ . We summarize the input/output alphabets of the MABC and TDBC protocols in Table III. In Section IV and V, we consider the case  $\mathcal{X}_i = \mathcal{Y}_i = \hat{\mathcal{Y}}_i = \mathbb{C}$ ,  $\forall i$ .

For a given block length  $n$ , it will be convenient to denote the transmission at time  $1 \leq k \leq n$  at node

<sup>3</sup> $\hat{\mathcal{Y}}_r$  is used in the CF TDBC protocol only.

$i$  by  $X_i^k$ , the reception at node  $i$  at time  $k$  by  $Y_i^k$ . Note that the distributions of  $X_i^k$  and  $Y_i^k$  depend on the value of  $k$ , e.g. for  $k \leq \Delta_{1,n} \cdot n$  we are in phase 1, for  $\Delta_{1,n}n < k \leq (\Delta_{1,n} + \Delta_{2,n})n$  we are in phase 2 and for  $(\Delta_{1,n} + \Delta_{2,n})n < k \leq n$  we are in phase 3 (in TDBC protocols only). During phase  $\ell$  we use  $X_i^{(\ell)}$  to denote the random variable with alphabet  $\mathcal{X}_i$  and input distribution  $p^{(\ell)}(x_i)$ . It is also convenient to denote by  $X_S^k := \{X_i^k | i \in S\}$ , the set of transmissions by all nodes in the set  $S$  at time  $k$ , and by  $X_S^{(\ell)} := \{X_i^{(\ell)} | i \in S\}$ , a set of random variables with channel input distribution  $p^{(\ell)}(x_S)$  for phase  $\ell$ , where  $x_S := \{x_i | i \in S\}$ . Lower case letters  $x_i$  will denote instances of the upper case  $X_i$  which lie in the calligraphic alphabets  $\mathcal{X}_i$ . Boldface  $\mathbf{x}_i$  represents a vector indexed by time at node  $i$ . Finally, we denote  $\mathbf{x}_S := \{\mathbf{x}_i | i \in S\}$  as a set of vectors indexed by time. In this work,  $Q$  will denote a discrete time-sharing random variable with distribution  $p(q)$ .

In order to define bi-directional communication rates we must define the encoders, decoders and associated probability of errors. We define  $W_{S,T} := \{W_{i,j} | i \in S, j \in T, S, T \subset \mathcal{M}\}$ . For a block length  $n$ , let  $\overleftarrow{\mathbf{y}}_i^{(\ell)}$  denote the set of received signals at node  $i$  up until the end of phase  $\ell$ , i.e., until the end of time  $n \cdot \sum_{m=1}^{\ell} \Delta_{m,n}$ . Encoders and decoders are functions  $X_i^{(\ell)}(W_{\{i\},\mathcal{M}}, \overleftarrow{\mathbf{y}}_i^{(\ell)}) \in \mathcal{X}_i$  and  $\tilde{W}_{j,i}(\overleftarrow{\mathbf{y}}_i^{(L)}, W_{\{i\},\mathcal{M}})$  respectively for  $\ell \in \{1, 2, \dots, L\}$  and for our proposed schemes, we will obtain single letter bounds. We define error events  $E_{i,j} := \{W_{i,j} \neq \tilde{W}_{i,j}(\cdot)\}$  for decoding the message  $W_{i,j}$  at node  $j$  at the end of the block of length  $n$ , and  $E_{i,j}^{(\ell)}$  as the error event at node  $j$  in which node  $j$  attempts to decode  $w_i$  at the end of phase  $\ell$  using a joint typicality decoder.

Let  $A^{(\ell)}(UV)$  represent the set of  $\epsilon$ -typical  $(\mathbf{u}^{(\ell)}, \mathbf{v}^{(\ell)})$  sequences of length  $n \cdot \Delta_{\ell,n}$  according to the distributions  $U$  and  $V$  in phase  $\ell$ . The events  $D^{(\ell)}(\mathbf{u}, \mathbf{v}) := \{(\mathbf{u}^{(\ell)}, \mathbf{v}^{(\ell)}) \in A^{(\ell)}(UV)\}$ . In general, joint typicality is non-transitive. However, by using strong joint-typicality, and the fact that for the distributions of interest  $x \rightarrow y \rightarrow \hat{y}$ , we will be able to argue joint typicality between  $\mathbf{x}$  and  $\hat{\mathbf{y}}$  by the *Markov lemma* of Lemma 4.1 in [2] and the extended Markov lemma (Lemma 3 of [17], Remark 30 of [11]).

A set of rates  $R_{i,j}$  is said to be achievable for a protocol with phase durations  $\{\Delta_\ell\}$  if there exist encoders/decoders of block length  $n = 1, 2, \dots$  with both  $P[E_{i,j}] \rightarrow 0$  and  $\Delta_{\ell,n} \rightarrow \Delta_\ell$  as  $n \rightarrow \infty$  for all  $\ell$ . An achievable rate region (resp. capacity region) is the closure of a set of (resp. all) achievable rate tuples for fixed  $\{\Delta_\ell\}$ .

## B. Previous results

We use the following outer bounds and achievable rate regions of decode and forward protocols, derived in [10] for comparison purposes in Sections IV and V. We simply state the results here for completeness.

*Theorem 1:* (Outer bound) The capacity region of the bi-directional relay channel constrained to the MABC protocol is outer bounded by the union of

$$R_a \leq \min\{\Delta_1 I(X_a^{(1)}; Y_r^{(1)} | X_b^{(1)}, Q), \Delta_2 I(X_r^{(2)}; Y_b^{(2)} | Q)\} \quad (1)$$

$$R_b \leq \min\{\Delta_1 I(X_b^{(1)}; Y_r^{(1)} | X_a^{(1)}, Q), \Delta_2 I(X_r^{(2)}; Y_a^{(2)} | Q)\} \quad (2)$$

over all joint distributions  $p(q)p^{(1)}(x_a|q)p^{(1)}(x_b|q)p^{(2)}(x_r|q)$  with  $|\mathcal{Q}| \leq 4$  over the alphabet  $\mathcal{X}_a \times \mathcal{X}_b \times \mathcal{X}_r$ .  $\square$

*Theorem 2:* (Outer bound) The capacity region of the half-duplex bi-directional relay channel constrained to the TDBC protocol is outer bounded by

$$R_a \leq \min\{\Delta_1 I(X_a^{(1)}; Y_r^{(1)}, Y_b^{(1)} | Q), \Delta_1 I(X_a^{(1)}; Y_b^{(1)} | Q) + \Delta_3 I(X_r^{(3)}; Y_b^{(3)} | Q)\} \quad (3)$$

$$R_b \leq \min\{\Delta_2 I(X_b^{(2)}; Y_r^{(2)}, Y_a^{(2)} | Q), \Delta_2 I(X_b^{(2)}; Y_a^{(2)} | Q) + \Delta_3 I(X_r^{(3)}; Y_a^{(3)} | Q)\} \quad (4)$$

$$R_a + R_b \leq \Delta_1 I(X_a^{(1)}; Y_r^{(1)} | Q) + \Delta_2 I(X_b^{(2)}; Y_r^{(2)} | Q) \quad (5)$$

over all joint distributions  $p(q)p^{(1)}(x_a|q)p^{(2)}(x_b|q)p^{(3)}(x_r|q)$  with  $|\mathcal{Q}| \leq 5$  over the alphabet  $\mathcal{X}_a \times \mathcal{X}_b \times \mathcal{X}_r$ .  $\square$

*Theorem 3:* An achievable rate region for the half-duplex bi-directional relay channel with the MABC protocol is the closure of the set of all points  $(R_a, R_b)$  satisfying

$$R_a < \min\{\Delta_1 I(X_a^{(1)}; Y_r^{(1)} | X_b^{(1)}, Q), \Delta_2 I(X_r^{(2)}; Y_b^{(2)} | Q)\} \quad (6)$$

$$R_b < \min\{\Delta_1 I(X_b^{(1)}; Y_r^{(1)} | X_a^{(1)}, Q), \Delta_2 I(X_r^{(2)}; Y_a^{(2)} | Q)\} \quad (7)$$

$$R_a + R_b < \Delta_1 I(X_a^{(1)}, X_b^{(1)}; Y_r^{(1)} | Q) \quad (8)$$

over all joint distributions  $p(q)p^{(1)}(x_a|q)p^{(1)}(x_b|q)p^{(2)}(x_r|q)$  with  $|\mathcal{Q}| \leq 5$  over the alphabet  $\mathcal{X}_a \times \mathcal{X}_b \times \mathcal{X}_r$ .  $\square$

*Theorem 4:* An achievable rate region for the half-duplex bi-directional relay channel with the TDBC protocol is the closure of the set of all points  $(R_a, R_b)$  satisfying

$$R_a < \min\{\Delta_1 I(X_a^{(1)}; Y_r^{(1)} | Q), \Delta_1 I(X_a^{(1)}; Y_b^{(1)} | Q) + \Delta_3 I(X_r^{(3)}; Y_b^{(3)} | Q)\} \quad (9)$$

$$R_b < \min\{\Delta_2 I(X_b^{(2)}; Y_r^{(2)} | Q), \Delta_2 I(X_b^{(2)}; Y_a^{(2)} | Q) + \Delta_3 I(X_r^{(3)}; Y_a^{(3)} | Q)\} \quad (10)$$

over all joint distributions  $p(q)p^{(1)}(x_a|q)p^{(2)}(x_b|q)p^{(3)}(x_r|q)$  with  $|\mathcal{Q}| \leq 4$  over the alphabet  $\mathcal{X}_a \times \mathcal{X}_b \times \mathcal{X}_r$ .  $\square$



### C. Compress and Forward using two joint typicality decoders

In Compress and Forward protocols, unlike in Decode and Forward protocols, the relay node  $r$  does not decode the message  $w_a$  or  $w_b$ . Thus, network coding techniques such as the algebraic group operation  $w_a \oplus w_b$  used in [10] cannot be used to generate  $w_r$  for the current CF schemes. Instead, two jointly typical decoders at each node are used to decode  $w_r$ .

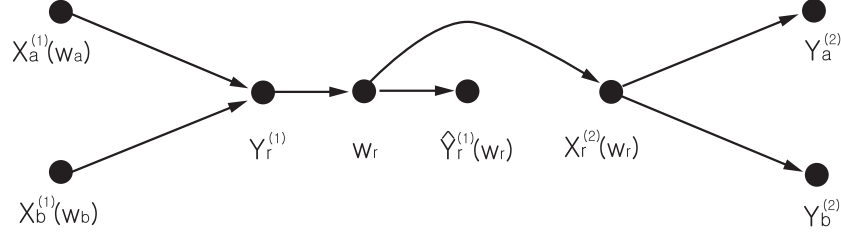


Fig. 1. The data flow in the compress and forward MABC protocol

To illustrate the decoding scheme, consider the decoder at node a which wishes to decode the relay message  $w_r$  in order to ultimately decode the desired message from node b,  $w_b$ . After phase 2, node a has the known sequences  $\mathbf{x}_a^{(1)}(w_a)$  and  $\mathbf{y}_a^{(2)}$ . Node a then finds the sets of all  $\hat{\mathbf{y}}_r^{(1)}(w_r)$  and  $\mathbf{x}_r^{(2)}(w_r)$  such that  $(\mathbf{x}_a^{(1)}(w_a), \hat{\mathbf{y}}_r^{(1)}(w_r))$  and  $(\mathbf{x}_r^{(2)}(w_r), \mathbf{y}_a^{(2)})$  are two pairs of jointly typical sequences, as shown in Fig. 1. Then node a decodes  $w_r$  correctly if there exists a unique  $w_r$  such that  $(\mathbf{x}_a^{(1)}(w_a), \hat{\mathbf{y}}_r^{(1)}(w_r)) \in A^{(1)}(X_a \hat{Y}_r)$  and  $(\mathbf{x}_r^{(2)}(w_r), \mathbf{y}_a^{(2)}) \in A^{(2)}(X_r Y_a)$  and declares a decoding error otherwise.

### III. ACHIEVABLE RATE REGIONS FOR COMPRESS AND FORWARD AND MIXED PROTOCOLS

In this Section we present three new achievable rate regions in Theorems 8, 12 and 14, and a slight improvement of [21] in Theorem 5. Theorems 5 and 8 are for two phase Multiple Access and Broadcast (MABC) protocols. In MABC protocols, nodes a and b transmit simultaneously as in a standard multiple-access channel in phase 1, the relay processes the received signal (either by decoding, amplifying, or compressing the signal as dictated by the protocol), and during phase 2 the relay broadcasts its signal to the two nodes. In Theorem 5 the relay simply uses a CF operation for both messages/directions, while in Theorem 8 the relay uses DF to transmit the  $w_a$  message while it uses CF to transmit the  $w_b$  message. The final two theorems 12 and 14 employ three phase Time Division Broadcast (TDBC) protocols. During the first phase, node a transmits while both the relay and node b receive its signal. During phase 2, node b is the sole transmitter while node a and the relay receive its transmission. After these phases,

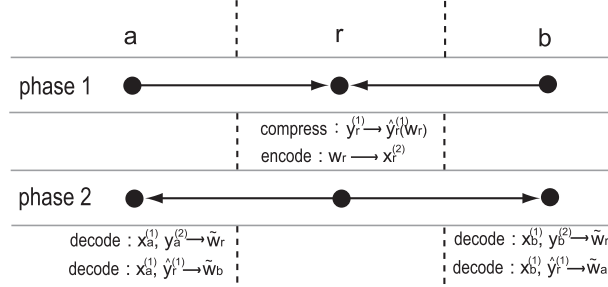


Fig. 2. The two-phase MABC protocol with a relay using a CF scheme.

the relay processes the received signals and is the sole transmitter during phase 3, during which it can aid nodes a and b to recover each others' messages. We now proceed to describe each protocol more precisely and present their respective achievable rate regions. Again, in Theorem 12 the relay uses CF when re-transmitting both messages, while in Theorem 14 the relay uses DF to transmit message  $w_a$  and CF to transmit message  $w_b$ . We now proceed to the main technical results of this work.

#### A. MABC Protocol

In the MABC Protocol, message  $w_a$  is communicated from node a to node b and message  $w_b$  is communicated from node b to node a with the help of the relay in two phases as shown in Fig. 2 and 3. During phase 1, nodes a and b simultaneously send independent messages  $w_a$  and  $w_b$  as codewords  $\mathbf{x}_a^{(1)}(w_a)$  and  $\mathbf{x}_b^{(1)}(w_b)$  to the relay, forming a classical multiple-access channel. Since we assume half-duplex nodes, neither a nor b can receive the message of the other during phase 1. The relay receives the signal  $\mathbf{y}_r^{(1)}$  according to  $p(y_r^{(1)} | x_a^{(1)}, x_b^{(1)})$ . Rather than attempting to decode message  $w_a$  and  $w_b$  (as in a DF scheme), it compresses the received  $\mathbf{y}_r^{(1)}$  into a signal  $\hat{\mathbf{y}}_r^{(1)}(w_r)$ . The index  $w_r$  is then mapped in a one-to-one fashion to the codeword  $\mathbf{x}_r^{(2)}(w_r)$  which is broadcast in phase 2 back to the relays. The challenge here is to determine the optimal compression strategy such that just enough information is carried back to the nodes to decode the opposite node's message. A key observation is that the nodes may use their own phase 1 transmitted messages as side-information in the decoding of phase 2 signals.

*Theorem 5:* An achievable rate region of the half-duplex bi-directional relay channel with the compress and forward MABC protocol is the closure of the set of all points  $(R_a, R_b)$  satisfying

$$R_a < \Delta_1 I(X_a^{(1)}; \hat{Y}_r^{(1)} | X_b^{(1)}, Q) \quad (11)$$

$$R_b < \Delta_1 I(X_b^{(1)}; \hat{Y}_r^{(1)} | X_a^{(1)}, Q) \quad (12)$$

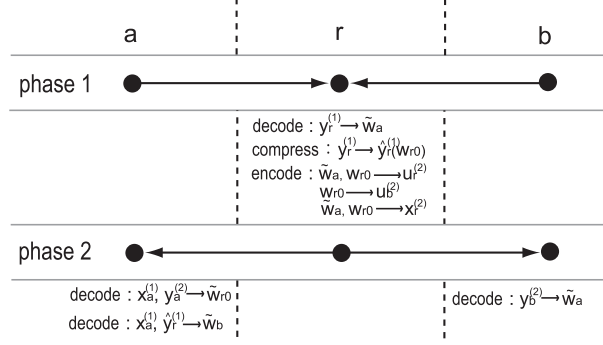


Fig. 3. The two-phase MABC protocol with a relay using a mixed scheme.

subject to

$$\Delta_1 I(Y_r^{(1)}; \hat{Y}_r^{(1)} | X_b^{(1)}, Q) < \Delta_2 I(X_r^{(2)}; Y_b^{(2)}) \quad (13)$$

$$\Delta_1 I(Y_r^{(1)}; \hat{Y}_r^{(1)} | X_a^{(1)}, Q) < \Delta_2 I(X_r^{(2)}; Y_a^{(2)}) \quad (14)$$

over all joint distributions,

$$p(q, x_a, x_b, x_r, y_a, y_b, y_r, \hat{y}_r) = p^{(1)}(q, x_a, x_b, y_r, \hat{y}_r | q) p^{(2)}(x_r, y_a, y_b) \quad (15)$$

where

$$p^{(1)}(q, x_a, x_b, y_r, \hat{y}_r) = p^{(1)}(q) p^{(1)}(x_a | q) p^{(1)}(x_b | q) p^{(1)}(y_r | x_a, x_b) p^{(1)}(\hat{y}_r | y_r, q) \quad (16)$$

$$p^{(2)}(x_r, y_a, y_b) = p^{(2)}(x_r) p^{(2)}(y_a, y_b | x_r) \quad (17)$$

with  $|\mathcal{Q}| \leq 4$  over the alphabet  $\mathcal{X}_a \times \mathcal{X}_b \times \mathcal{X}_r \times \mathcal{Y}_a \times \mathcal{Y}_b \times \mathcal{Y}_r \times \hat{\mathcal{Y}}_r$ .  $\square$

*Remark 6:* The bound of Theorem 5 is essentially derived in [21]; equation (16) is a slight extension, as we use  $p^{(1)}(\hat{y}_r | y_r, q)$  instead of  $p^{(1)}(\hat{y}_r | y_r)$ , i.e., in [21] the codewords  $\hat{\mathbf{y}}_r^{(1)}$  are generated according to  $p^{(1)}(\hat{y}_r) = \sum p^{(1)}(y_r) p^{(1)}(\hat{y}_r | y_r)$ , while in (16) the distribution space  $p^{(1)}(\hat{y}_r | q) = \sum p^{(1)}(y_r) p^{(1)}(\hat{y}_r | y_r, q)$  is larger. By conditioning on  $q$ , one can “fine-tune” the distribution of  $\hat{\mathbf{y}}_r^{(1)}$  for each given  $q$  and the left side of (13) and (14) can be reduced. This is because the distributions of  $X_a^{(1)}$  and  $X_b^{(1)}$ , and hence  $Y_r^{(1)}$ , depend on  $q$ . For example, let  $p^{(1)}(q = 1) = \alpha_n$  and  $p^{(1)}(q = 2) = 1 - \alpha_n$ , where  $0 < \alpha_n < 1$ . For  $q = 1$  we optimize  $p^{(1)}(\hat{y}_r | 1)$  and generate  $(\alpha_n \Delta_{1,n} \cdot n)$ -length sequence  $\hat{\mathbf{y}}_r^{(1),1}(w_{r1})$ ,  $w_{r1} \in \{0, 1, \dots, \lfloor 2^{nR_{r1}} \rfloor\}$ , where  $R_{r1} = \alpha_n \Delta_{1,n} (I(Y_r^{(1)}; \hat{Y}_r^{(1)} | q = 1) + \epsilon)$ . Likewise, we generate  $\hat{\mathbf{y}}_r^{(1),2}(w_{r2})$  for  $q = 2$ . To compress  $\mathbf{y}_r^{(1)}$  to  $\hat{\mathbf{y}}_r^{(1)}$ , we construct  $\mathbf{y}_r^{(1)} = (\mathbf{y}_r^{(1),1}, \mathbf{y}_r^{(1),2})$  and  $\hat{\mathbf{y}}_r^{(1)} = (\hat{\mathbf{y}}_r^{(1),1}, \hat{\mathbf{y}}_r^{(1),2})$  and choose  $w_r = (w_{r1}, w_{r2})$  if both  $(\mathbf{y}_r^{(1),1}, \hat{\mathbf{y}}_r^{(1),1}(w_{r1}))$  and  $(\mathbf{y}_r^{(1),2}, \hat{\mathbf{y}}_r^{(1),2}(w_{r2}))$  are jointly typical. Then the

rate  $R_r = R_{r1} + R_{r2} = \Delta_1 I(Y_r^{(1)}; \hat{Y}_r^{(1)} | Q)$ . However, if one generated  $\hat{y}_r^{(1)}$  from the distribution  $p^{(1)}(\hat{y}_r)$  then  $R_r = \Delta_1 I(Y_r^{(1)}; \hat{Y}_r^{(1)}) \geq \Delta_1 I(Y_r^{(1)}; \hat{Y}_r^{(1)} | Q)$  with strict inequality except in degenerate cases.

*Remark 7:* Strong typicality is required for the proof of Theorem 5 in order to apply the Markov lemma to  $(X_a^{(1)}, X_b^{(1)}) \rightarrow Y_r^{(1)} \rightarrow \hat{Y}_r^{(1)}$  for each given  $q$ . Since strong typicality is defined for discrete alphabets, Theorem 5 cannot be directly extended to continuous alphabets. However, the extended Markov lemma (see Remark 30 of [11] as well as Lemma 3 of [17]) shows that for Gaussian distributions, the Markov lemma still applies.

The previous theorem assumed the relay used a CF scheme for both messages  $w_a$  and  $w_b$ . However, in the event of asymmetric channel gains between the two nodes at the relay, it may be beneficial to have the stronger channel use a DF scheme while the weaker channel uses a CF scheme since decoding may not be possible. We next consider a mixed MABC strategy in which phase 1 is still a multiple access channel and phase 2 is still a broadcast channel. However, the relay uses a novel strategy in determining its phase 2 codeword  $\mathbf{x}_r^{(2)}$ . That is, the relay operates such that the  $a \rightarrow r \rightarrow b$  link uses decode and forward while  $a \leftarrow r \leftarrow b$  link uses compress and forward. Furthermore, the relay applies a Gelfand-Pinsker coding scheme to protect  $w_a$  in the  $r \rightarrow b$  link. In this case, an achievable rate region is given by Theorem 8.

*Theorem 8:* An achievable rate region of the half-duplex bi-directional relay channel with the mixed forward MABC protocol, where  $a \rightarrow b$  link uses decode and forward and  $b \rightarrow a$  link uses compress and forward, is the closure of the set of all points  $(R_a, R_b)$  satisfying

$$R_a < \min \left\{ \Delta_1 I(X_a^{(1)}; Y_r^{(1)} | Q), \Delta_2 I(U_r^{(2)}; Y_b^{(2)} | Q) - \Delta_2 I(U_r^{(2)}; U_b^{(2)} | Q) \right\} \quad (18)$$

$$R_b < \Delta_1 I(X_b^{(1)}; \hat{Y}_r^{(1)} | X_a^{(1)}, Q) \quad (19)$$

subject to

$$\Delta_1 I(Y_r^{(1)}; \hat{Y}_r^{(1)} | X_a^{(1)}, Q) < \min \{ \Delta_2 I(U_r^{(2)}, U_b^{(2)}; Y_a^{(2)} | Q), \Delta_2 I(U_b^{(2)}; U_r^{(2)}, Y_a^{(2)} | Q) \} \quad (20)$$

over all joint distributions,

$$p(q, x_a, x_b, x_r, u_a, u_b, u_r, y_a, y_b, y_r, \hat{y}_r) = p(q) p^{(1)}(x_a, x_b, y_r, \hat{y}_r | q) p^{(2)}(u_b, u_r, x_r, y_a, y_b | q) \quad (21)$$

where

$$p^{(1)}(x_a, x_b, y_r, \hat{y}_r | q) = p^{(1)}(x_a | q) p^{(1)}(x_b | q) p^{(1)}(y_r | x_a, x_b) p^{(1)}(\hat{y}_r | y_r, q) \quad (22)$$

$$p^{(2)}(u_b, u_r, x_r, y_a, y_b | q) = p^{(2)}(u_b, u_r | q) p^{(2)}(x_r | u_b, u_r, q) p^{(2)}(y_a, y_b | x_r) \quad (23)$$

with  $|Q| \leq 7$  over the alphabet  $\mathcal{X}_a \times \mathcal{X}_b \times \mathcal{X}_r \times \mathcal{U}_b \times \mathcal{U}_r \times \mathcal{Y}_a \times \mathcal{Y}_b \times \mathcal{Y}_r \times \hat{\mathcal{Y}}_r$ .  $\square$

*Remark 9:* In the second phase, the relay broadcasts the received signals from the first phase. In contrast to the DF and CF schemes, one of the terminal nodes (in this case, node a) has perfect information of interference at node a, while there remains unknown interference at the other side (at node b). We use a Gel'fand-Pinsker coding scheme [7], [6] for the link  $r \rightarrow b$  which yields the second term of (18). From the side information  $w_a$  available at node a, node a is able to reduce the interference, yielding (20).

*Remark 10:* If we apply the achievable bound to the Gaussian noise channel without fading with Costa's setup in [3] with  $|Q| = 1$  we have in phase two:

$$U_r^{(2)} = V_r^{(2)} + \alpha U_b^{(2)} \quad (24)$$

$$Y_a^{(2)} = V_r^{(2)} + U_b^{(2)} + Z_a^{(2)} \quad (25)$$

$$Y_b^{(2)} = V_r^{(2)} + U_b^{(2)} + Z_b^{(2)} \quad (26)$$

where  $Z_a^{(2)} \sim \mathcal{CN}(0, N_a)$ ,  $Z_b^{(2)} \sim \mathcal{CN}(0, N_b)$ ,  $U_b^{(2)} \sim \mathcal{CN}(0, P_{U_r})$ ,  $V_r^{(2)} \sim \mathcal{CN}(0, P_{V_r})$ , and  $V_r^{(2)}$ ,  $U_b^{(2)}$  are independent and  $P_{U_r} + P_{V_r} = P_r$ .  $V_r^{(2)}$  is an intermediate random variable generated according to  $\mathcal{CN}(0, P_r)$  which contains the information to be transmitted from r to a. Then from (18) and (20) the achievable rate of link  $r \rightarrow a$ ,  $R_{ra}$  (resp.  $R_{rb}$  for  $r \rightarrow b$ ) is :

$$R_{ra} = \min \left\{ \log_2 \left( 1 + \frac{P_r}{N_a} \right), \log_2 \left( \frac{P_{V_r} P_{U_b} (1 - \alpha)^2 + N_a (P_{V_r} + \alpha^2 P_{U_b})}{N_a P_{V_r}} \right) \right\} \quad (27)$$

$$R_{rb} = \log_2 \left( \frac{P_{V_r} (P_{V_r} + P_{U_b} + N_b)}{P_{V_r} P_{U_b} (1 - \alpha)^2 + N_b (P_{V_r} + \alpha^2 P_{U_b})} \right) \quad (28)$$

*Proof: Random code generation:* For simplicity of exposition, we take  $|Q| = 1$ .

1) Phase 1: Generate random  $(n \cdot \Delta_{1,n})$ -length sequences

- $\mathbf{x}_a^{(1)}(w_a)$  i.i.d. with  $p^{(1)}(x_a)$ ,  $w_a \in \mathcal{S}_a = \{0, 1, \dots, \lfloor 2^{nR_a} \rfloor - 1\}$
- $\mathbf{x}_b^{(1)}(w_b)$  i.i.d. with  $p^{(1)}(x_b)$ ,  $w_b \in \mathcal{S}_b = \{0, 1, \dots, \lfloor 2^{nR_b} \rfloor - 1\}$
- $\hat{\mathbf{y}}_r^{(1)}(w_{r0})$  i.i.d. with  $p^{(1)}(\hat{y}_r) = \sum_{y_r} p^{(1)}(y_r) p^{(1)}(\hat{y}_r | y_r)$ ,  $w_{r0} \in \{0, 1, \dots, \lfloor 2^{nR_{r0}} \rfloor - 1\} := \mathcal{S}_{r0}$

2) Phase 2: Generate random  $(n \cdot \Delta_{2,n})$ -length sequences

- $\mathbf{u}_r^{(2)}(w_r)$  i.i.d. with  $p^{(2)}(u_r)$ ,  $w_r \in \{0, 1, \dots, \lfloor 2^{nR_r} \rfloor - 1\} := \mathcal{S}_r$
- $\mathbf{u}_b^{(2)}(w_{r0})$  i.i.d. with  $p^{(2)}(u_b)$ ,  $w_{r0} \in \mathcal{S}_{r0}$

and define bin  $B_i := \{w_r | w_r \in [(i-1) \cdot \lfloor 2^{n(R_r - R_a)} \rfloor + 1, i \cdot \lfloor 2^{n(R_r - R_a)} \rfloor]\}$  for  $i \in \mathcal{S}_a$ .

*Encoding:* During phase 1, the encoders of node a and b send the codewords  $\mathbf{x}_a^{(1)}(w_a)$  and  $\mathbf{x}_b^{(1)}(w_b)$  respectively. At the end of phase 1, relay r decodes  $\tilde{w}_a$  and maps  $\mathbf{y}_r^{(1)}$  to a message index  $w_{r0}$  if there exists a  $w_{r0}$  such that  $(\mathbf{y}_r^{(1)}, \hat{\mathbf{y}}_r^{(1)}(w_{r0})) \in A^{(1)}(Y_r \hat{Y}_r)$ . Such a  $w_{r0}$  exists with high probability if

$$R_{r0} = \Delta_{1,n} I(Y_r^{(1)}; \hat{Y}_r^{(1)}) + \epsilon \quad (29)$$

and  $n$  is sufficiently large. We choose

$$R_r = \Delta_{2,n}I(U_r^{(2)}; Y_b^{(2)}) - 4\epsilon. \quad (30)$$

To choose  $w_r$ , the relay first selects the bin  $B_{\tilde{w}_a}$  and then it searches for the minimum  $w_r \in B_{\tilde{w}_a}$  such that  $(\mathbf{u}_r^{(2)}(w_r), \mathbf{u}_b^{(2)}(w_{r0})) \in A^{(2)}(U_r U_b)$ . This ensures uniqueness of  $w_r$  if such a  $w_r$  exists, i.e.,  $w_r$  is a function of  $(w_a, w_{r0})$ . Such a  $w_r$  exists with high probability if

$$|B_{\tilde{w}_a}| \geq 2^{n(\Delta_{2,n}I(U_r^{(2)}; U_b^{(2)}) + \epsilon)}. \quad (31)$$

Since  $|B_i| = 2^{n(R_r - R_a)}$ ,  $\forall i \in \mathcal{S}_a$ , this condition is equivalent to

$$R_a < \Delta_{2,n}I(U_r^{(2)}; Y_b^{(2)}) - \Delta_{2,n}I(U_r^{(2)}; U_b^{(2)}) - 5\epsilon. \quad (32)$$

The relay then sends  $\mathbf{x}_r^{(2)}$  randomly generated i.i.d. according to  $p^{(2)}(x_r|u_r, u_b)$  with  $\mathbf{u}_r^{(2)}(w_r)$  and  $\mathbf{u}_b^{(2)}(w_{r0})$  during phase 2.

*Decoding:* Node a estimates  $\tilde{w}_{r0}$  after phase 2 using jointly typical decoding. First, since a knows  $w_a$ , it can reduce the cardinality of  $w_r$  to  $\lfloor 2^{n(R_r - R_a)} \rfloor$ . Furthermore, it forms two sets of  $\tilde{w}_{r0}$  based on typical sequences,  $\{\tilde{w}_{r0} | (\mathbf{x}_a^{(1)}(w_a), \hat{\mathbf{y}}_r^{(1)}(\tilde{w}_{r0})) \in A^{(1)}(X_a \hat{Y}_r)\}$  and  $\{\tilde{w}_{r0} | (\mathbf{u}_r^{(2)}(\tilde{w}_r), \mathbf{u}_b^{(2)}(\tilde{w}_{r0}), \mathbf{y}_a^{(2)}) \in A^{(2)}(U_r U_b Y_a), \tilde{w}_r \in B_{w_a}\}$ . After decoding  $\tilde{w}_{r0}$  (which is a success if there is a single common element in both of the previous sets), a decodes  $\tilde{w}_b$  using jointly typical decoding of the sequence  $(\mathbf{x}_a^{(1)}, \mathbf{x}_b^{(1)}, \hat{\mathbf{y}}_r^{(1)})$ . Node b decodes  $\tilde{w}_r$  after phase 2 and from the bin index of  $\tilde{w}_r$  it estimates  $\tilde{w}_a$ .

*Error analysis:* By the union bound,

$$P[E_{a,b}] \leq P[E_{a,r}^{(1)} \cup E_{r,b}^{(2)}] \quad (33)$$

$$\leq P[E_{a,r}^{(1)}] + P[E_{r,b}^{(2)} | \bar{E}_{a,r}^{(1)}] \quad (34)$$

$$P[E_{b,a}] \leq P[E_{r,a}^{(2)} \cup E_{b,a}^{(2)}] \quad (35)$$

$$\leq P[E_{r,a}^{(2)}] + P[E_{b,a}^{(2)} | \bar{E}_{r,a}^{(2)}] \quad (36)$$

We define error events in each phase as follows:

- 1)  $E_{a,r}^{(1)} = E_{a,r}^{(1),1} \cup E_{a,r}^{(1),2}$ .  
 $E_{a,r}^{(1),1} : (\mathbf{x}_a^{(1)}(w_a), \mathbf{y}_r^{(1)}) \notin A^{(1)}(X_a Y_r)$ .  
 $E_{a,r}^{(1),2} : \text{there exists } \tilde{w}_a \neq w_a \text{ such that } (\mathbf{x}_a^{(1)}(\tilde{w}_a), \mathbf{y}_r^{(1)}) \in A^{(1)}(X_a Y_r)$ .
- 2)  $E_{r,b}^{(2)} = E_{r,b}^{(2),1} \cup E_{r,b}^{(2),2} \cup E_{r,b}^{(2),3} \cup E_{r,b}^{(2),4}$ .  
 $E_{r,b}^{(2),1} : \text{there does not exist a } w_{r0} \text{ such that } (\mathbf{y}_r^{(1)}, \hat{\mathbf{y}}_r^{(1)}(w_{r0})) \in A^{(1)}(Y_r \hat{Y}_r)$ .  
 $E_{r,b}^{(2),2} : \text{there does not exist a } w_r \in B_{w_a} \text{ such that } (\mathbf{u}_r^{(2)}(w_r), \mathbf{u}_b^{(2)}(w_{r0})) \in A^{(2)}(U_r U_b)$ .

- $E_{r,b}^{(2),3} : (\mathbf{u}_r^{(2)}(w_r), \mathbf{y}_b^{(2)}) \notin A^{(2)}(U_r Y_b).$   
 $E_{r,b}^{(2),4} : \text{there exists } \tilde{w}_r \neq w_r \text{ such that } (\mathbf{u}_r^{(2)}(\tilde{w}_r), \mathbf{y}_b^{(2)}) \in A^{(2)}(U_r Y_b).$   
 3)  $E_{r,a}^{(2)} = E_{r,a}^{(2),1} \cup E_{r,a}^{(2),2} \cup E_{r,a}^{(2),3} \cup E_{r,a}^{(2),4} \cup E_{r,a}^{(2),5} \cup E_{r,a}^{(2),6}.$   
 $E_{r,a}^{(2),1} : \text{there does not exist a } w_{r0} \text{ such that } (\mathbf{y}_r^{(1)}, \hat{\mathbf{y}}_r^{(1)}(w_{r0})) \in A^{(1)}(Y_r \hat{Y}_r).$   
 $E_{r,a}^{(2),2} : \text{there does not exist a } w_r \in B_{w_a} \text{ such that } (\mathbf{u}_r^{(2)}(w_r), \mathbf{u}_b^{(2)}(w_{r0})) \in A^{(2)}(U_r U_b).$   
 $E_{r,a}^{(2),3} : (\mathbf{x}_a^{(1)}(w_a), \hat{\mathbf{y}}_r^{(1)}(w_{r0})) \notin A^{(1)}(X_a \hat{Y}_r).$   
 $E_{r,a}^{(2),4} : (\mathbf{u}_r^{(2)}(w_r), \mathbf{u}_b^{(2)}(w_{r0}), \mathbf{y}_a^{(2)}) \notin A^{(2)}(U_r U_b Y_a).$   
 $E_{r,a}^{(2),5} : \text{there exists } (\tilde{w}_r, \tilde{w}_{r0}) \text{ where } \tilde{w}_r \neq w_r \text{ and } \tilde{w}_{r0} \neq w_{r0} \text{ such that } (\mathbf{u}_r^{(2)}(\tilde{w}_r), \mathbf{u}_b^{(2)}(\tilde{w}_{r0}), \mathbf{y}_a^{(2)}) \in A^{(2)}(U_r U_b Y_a) \text{ and } (\mathbf{x}_a^{(1)}(w_a), \hat{\mathbf{y}}_r^{(1)}(\tilde{w}_{r0})) \in A^{(1)}(X_a \hat{Y}_r). \text{ Recall, } w_r \text{ is uniquely specified by } (w_a, w_{r0}). \text{ Hence for a given } w_a, \text{ there are at most } 2^{nR_{r0}} \text{ such } (\tilde{w}_r, \tilde{w}_{r0}) \text{ pairs.}$   
 $E_{r,a}^{(2),6} : \text{there exists } \tilde{w}_{r0} \neq w_{r0} \text{ such that } (\mathbf{u}_r^{(2)}(w_r), \mathbf{u}_b^{(2)}(\tilde{w}_{r0}), \mathbf{y}_a^{(2)}) \in A^{(2)}(U_r U_b Y_a),$   
 $(\mathbf{x}_a^{(1)}(w_a), \hat{\mathbf{y}}_r^{(1)}(\tilde{w}_{r0})) \in A^{(1)}(X_a \hat{Y}_r).$   
 4)  $E_{b,a}^{(2)} = E_{b,a}^{(2),1} \cup E_{b,a}^{(2),2}.$   
 $E_{b,a}^{(2),1} : (\mathbf{x}_a^{(1)}(w_a), \mathbf{x}_b^{(1)}(w_b), \hat{\mathbf{y}}_r^{(1)}(w_{r0})) \notin A^{(1)}(X_a X_b \hat{Y}_r).$   
 $E_{b,a}^{(2),2} : \text{there exists } \tilde{w}_b \neq w_b \text{ such that } (\mathbf{x}_a^{(1)}(w_a), \mathbf{x}_b^{(1)}(\tilde{w}_b), \hat{\mathbf{y}}_r^{(1)}(w_{r0})) \in A^{(1)}(X_a X_b \hat{Y}_r).$

Then:

$$P[E_{a,r}^{(1)}] \leq P[E_{a,r}^{(1),1}] + P[E_{a,r}^{(1),2}] \quad (37)$$

$$= P[\bar{D}^{(1)}(\mathbf{x}_a(w_a), \mathbf{y}_r)] + P[\cup_{\tilde{w}_a \neq w_a} D^{(1)}(\mathbf{x}_a(\tilde{w}_a), \mathbf{y}_r)] \quad (38)$$

$$\leq \epsilon + 2^{n(R_a - \Delta_{1,n} I(X_a^{(1)}; Y_r^{(1)})) + 3\epsilon} \quad (39)$$

$$P[E_{r,b}^{(2)} | \bar{E}_{a,r}^{(1)}] \leq P[E_{r,b}^{(2),1} | \bar{E}_{a,r}^{(1)}] + P[E_{r,b}^{(2),2} | \bar{E}_{a,r}^{(1)}] + P[E_{r,b}^{(2),3} | \bar{E}_{a,r}^{(1)}] + P[E_{r,b}^{(2),4} | \bar{E}_{a,r}^{(1)}] \quad (40)$$

$$\leq 2\epsilon + P[\bar{D}^{(2)}(\mathbf{u}_r(w_r), \mathbf{y}_b)] + P[\cup_{\tilde{w}_r \neq w_r} D^{(2)}(\mathbf{u}_r(\tilde{w}_r), \mathbf{y}_b)] \quad (41)$$

$$\leq 3\epsilon + 2^{n(R_r - \Delta_{2,n} I(U_r^{(2)}; Y_b^{(2)})) + 3\epsilon} \quad (42)$$

In (41),  $P[E_{r,b}^{(2),1} | \bar{E}_{a,r}^{(1)}]$  and  $P[E_{r,b}^{(2),2} | \bar{E}_{a,r}^{(1)}]$  are less than  $\epsilon$  due to (29) and (32), respectively. Furthermore,

$$P[E_{r,a}^{(2)}] \leq P[E_{r,a}^{(2),1}] + P[E_{r,a}^{(2),2}] + P[E_{r,a}^{(2),3}] + P[E_{r,a}^{(2),4}] + P[E_{r,a}^{(2),5}] + P[E_{r,a}^{(2),6}] \quad (43)$$

$$\leq 2\epsilon + P[\bar{D}^{(1)}(\mathbf{x}_a(w_a), \hat{\mathbf{y}}_r(w_{r0}))] + P[\bar{D}^{(2)}(\mathbf{u}_r(w_r), \mathbf{u}_b(w_{r0}), \mathbf{y}_a)] +$$

$$P[\cup_{\substack{\tilde{w}_r \neq w_r \\ \tilde{w}_{r0} \neq w_{r0}}} D^{(2)}(\mathbf{u}_r(\tilde{w}_r), \mathbf{u}_b(\tilde{w}_{r0}), \mathbf{y}_a), D^{(1)}(\mathbf{x}_a(w_a), \hat{\mathbf{y}}_r(\tilde{w}_{r0})))] +$$

$$P[\cup_{\substack{\tilde{w}_r = w_r \\ \tilde{w}_{r0} \neq w_{r0}}} D^{(2)}(\mathbf{u}_r(w_r), \mathbf{u}_b(\tilde{w}_{r0}), \mathbf{y}_a), D^{(1)}(\mathbf{x}_a(w_a), \hat{\mathbf{y}}_r(\tilde{w}_{r0})))] \quad (44)$$

$$\leq 4\epsilon + 2^{n(R_{r0} - \Delta_{2,n} I(U_r^{(2)}, U_b^{(2)}; Y_a^{(2)}) - \Delta_{1,n} I(\hat{Y}_r^{(1)}; X_a^{(1)}) + 7\epsilon)} +$$

$$2^{n(R_{r0} - \Delta_{2,n} I(U_b^{(2)}; U_r^{(2)}, Y_a^{(2)}) - \Delta_{1,n} I(\hat{Y}_r^{(1)}; X_a^{(1)}) + 7\epsilon)} \quad (45)$$

$$P[E_{b,a}^{(2)} | \bar{E}_{r,a}^{(2)}] \leq P[E_{b,a}^{(2),1} | \bar{E}_{r,a}^{(2)}] + P[E_{b,a}^{(2),2} | \bar{E}_{r,a}^{(2)}] \quad (46)$$

$$= P[\bar{D}^{(1)}(\mathbf{x}_a(w_a), \mathbf{x}_b(w_b), \hat{\mathbf{y}}_r(w_{r0}))] + P[\cup_{\tilde{w}_b \neq w_b} D^{(1)}(\mathbf{x}_a(w_a), \mathbf{x}_b(\tilde{w}_b), \hat{\mathbf{y}}_r(w_{r0}))] \quad (47)$$

$$\leq \epsilon + 2^{n(R_b - \Delta_{1,n} I(X_b^{(1)}; \hat{Y}_r^{(1)} | X_a^{(1)}) + 4\epsilon)} \quad (48)$$

In (44),  $P[E_{r,a}^{(2),1}]$  and  $P[E_{r,a}^{(2),2}]$  are less than  $\epsilon$  due to (29) and (32), respectively. In (45),  $P[\bar{D}^{(1)}(\mathbf{x}_a(w_a), \hat{\mathbf{y}}_r(w_{r0}))]$  is less than  $\epsilon$  by the Markov lemma and the total cardinality of the case ( $\tilde{w}_r \neq w_r$  and  $\tilde{w}_{r0} \neq w_{r0}$ ) is bounded by  $2^{nR_{r0}}$  since  $\tilde{w}_r$  is uniquely specified if  $(w_a, \tilde{w}_{r0})$  is given.

Since  $\epsilon > 0$  is arbitrary, the conditions of Theorem 8, (30) and the AEP property guarantee that the right hand sides of (39), (42), (45) and (48) corresponding to the first term of (18), (30), (20) and (19) respectively vanish as  $n \rightarrow \infty$ . By the Carathéodory theorem in [9], it is sufficient to restrict  $|\mathcal{Q}| \leq 7$ . ■

The mixed MABC region in Theorem 8 is outer bounded by the DF MABC region in Theorem 3. In the mixed MABC protocol, the relay  $r$  has to be able to decode  $w_a$  correctly after phase 1 without any information about  $w_b$ . If node  $a$  can decode  $w_b$  from a compressed version of  $y_r$  and knowledge of  $w_a$ , then by the information processing inequality, node  $r$  can decode  $w_b$  from  $y_r$  and  $w_a$ .

*Theorem 11:* The achievable rate region of the mixed MABC protocol of Theorem 8 is outer bounded by the achievable rate region of the DF MABC protocol of Theorem 3. □

*Proof:* If  $R_a$  and  $R_b$  lies in the region of the mixed MABC protocol of Theorem 8 for a given distribution  $p^{(1)}(x_a)p^{(1)}(x_b)p^{(1)}(y_r|x_a, x_b)p^{(1)}(\hat{y}_r|y_r)p^{(2)}(x_r)p^{(2)}(y_a, y_b|x_r)$  then:

$$R_a < \Delta_1 I(X_a^{(1)}; Y_r^{(1)}) \leq \Delta_1 I(X_a^{(1)}; Y_r^{(1)} | X_b^{(1)}) \quad (49)$$

$$R_a < \Delta_2 I(U_r^{(2)}; Y_b^{(2)}) - \Delta_2 I(U_r^{(2)}; U_b^{(2)}) \leq \Delta_2 I(X_r^{(2)}; Y_b^{(2)}) \quad (50)$$

$$R_b < \Delta_1 I(X_b^{(1)}; \hat{Y}_r^{(1)} | X_a^{(1)}) \leq \Delta_1 I(X_b^{(1)}; Y_r^{(1)} | X_a^{(1)}) \quad (51)$$

$$\begin{aligned} R_b &< \Delta_1 I(X_b^{(1)}; \hat{Y}_r^{(1)} | X_a^{(1)}) \leq \Delta_1 I(Y_r^{(1)}; \hat{Y}_r^{(1)} | X_a^{(1)}) \\ &\leq \Delta_2 I(U_r^{(2)}, U_b^{(2)}; Y_a^{(2)}) \leq \Delta_2 I(X_r^{(2)}; Y_a^{(2)}) \end{aligned} \quad (52)$$

$$R_a + R_b < \Delta_1 I(X_a^{(1)}; Y_r^{(1)}) + \Delta_1 I(X_b^{(1)}; Y_r^{(1)} | X_a^{(1)}) \leq \Delta_1 I(X_a^{(1)}, X_b^{(1)}; Y_r^{(1)}) \quad (53)$$

(50) and (52) are from the Markov process  $(U_r^{(2)}, U_b^{(2)}) \rightarrow X_r^{(2)} \rightarrow (Y_a^{(2)}, Y_b^{(2)})$ . From (49) – (53),  $R_a$  and  $R_b$  are in the region of the DF MABC protocol in the Theorem 3. Therefore, every point  $(R_a, R_b)$  represented by a convex combination of distributions (or using  $\mathcal{Q}$ ) in the Mixed MABC protocol is also achievable with the DF MABC protocol. ■



From Theorem 11, the mixed MABC protocol does not achieve any rate pairs which cannot be achieved by the DF MABC protocol. However, its possible benefit lies in that it only requires the relay  $r$  to possess *one* of the codebooks of  $a$  and  $b$ . Therefore, in the event that relay  $r$  has one of the codebooks of the terminal nodes, by employing the Mixed MABC protocol one can achieve a rate region which outperforms that of the CF MABC protocol. In practice, if there are many terminal and relay nodes a relay may have some but not all codebooks of the terminal nodes. If the relay has full codebook knowledge then DF is possible, otherwise CF (or AF) may be more appropriate.

### B. TDBC Protocol

The Time Division Broadcast (TDBC) protocol consists of three phases rather than the two seen in the MABC protocol. The MABC protocol takes advantage of the various gains provided by multiple-access schemes by having both nodes  $a$  and  $b$  transmit during phase 1. However, the possible direct links between nodes  $a$  and  $b$  are not exploited. The TDBC protocol aims to exploit the direct link by having the nodes combine the signals received on the direct link and through the relay node, that is, exploit the side-information available at the decoders.

The TDBC protocol consists of three phases, as illustrated in Fig. 4 and 5. During phase 1, node  $a$  is the sole node to transmit, while both the relay and node  $b$  receive this transmission. During phase 2, node  $b$  transmits while the relay and node  $a$  receive. After phase 2, the relay processes the signals received during the first 2 phases and proceeds to broadcast to nodes  $a$  and  $b$  during the third phase.

In the CF TDBC protocol, we use two different broadcasting schemes in the last phase. For this reason, we divide phase 3 into two sub-phases. In the first relay-broadcast phase, we use Marton's broadcast scheme of [13], in which two different messages are transmitted to the two receivers. In this scheme, neither receiver uses side information ( $w_a$  at node  $a$  and  $w_b$  at node  $b$ ) to decode the messages. In the second relay-broadcasting phase, we assume a compound channel, i.e., a common message is transmitted to the two receivers which have different side information.

For convenience of analysis, we denote the first part of the relay-broadcast phase as phase 3 and the second as phase 4.

*Theorem 12:* An achievable rate region of the half-duplex bi-directional relay channel with the compress and forward TDBC protocol is the closure of the set of all points  $(R_a, R_b)$  satisfying

$$R_a < \Delta_1 I(X_a^{(1)}; \hat{Y}_r^{(1)}, Y_b^{(1)} | Q) \quad (54)$$

$$R_b < \Delta_2 I(X_b^{(2)}; \hat{Y}_r^{(2)}, Y_a^{(2)} | Q) \quad (55)$$

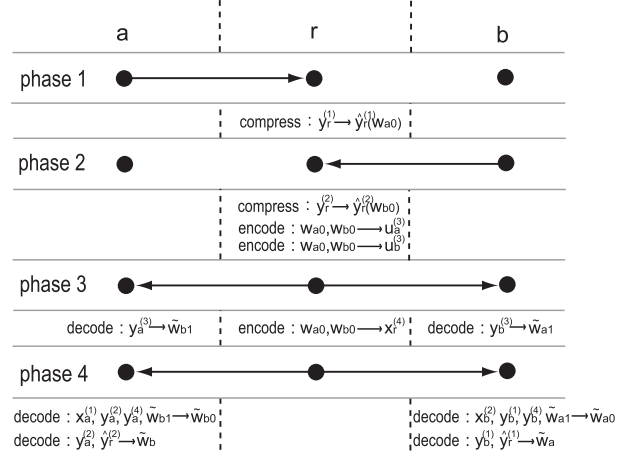


Fig. 4. The three-phase TDBC protocol with a relay using a CF scheme.

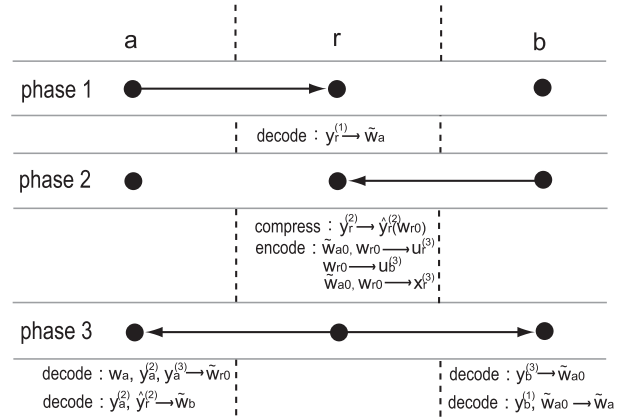


Fig. 5. The three-phase TDBC protocol with a relay using a mixed scheme.

subject to

$$\alpha_a \Delta_1 I(Y_r^{(1)}; \hat{Y}_r^{(1)} | Q) < \Delta_3 I(U_a^{(3)}; Y_b^{(3)} | Q) \quad (56)$$

$$\alpha_b \Delta_2 I(Y_r^{(2)}; \hat{Y}_r^{(2)} | Q) < \Delta_3 I(U_b^{(3)}; Y_a^{(3)} | Q) \quad (57)$$

$$\alpha_a \Delta_1 I(Y_r^{(1)}; \hat{Y}_r^{(1)} | Q) + \alpha_b \Delta_2 I(Y_r^{(2)}; \hat{Y}_r^{(2)} | Q) < \Delta_3 I(U_a^{(3)}; Y_b^{(3)} | Q) + \Delta_3 I(U_b^{(3)}; Y_a^{(3)} | Q) - \Delta_3 I(U_a^{(3)}; U_b^{(3)} | Q) \quad (58)$$

$$(1 - \alpha_a) \Delta_1 I(Y_r^{(1)}; \hat{Y}_r^{(1)} | Q) + \Delta_2 I(Y_r^{(2)}; \hat{Y}_r^{(2)} | X_b^{(2)}, Q) < \Delta_4 I(X_r^{(4)}; Y_b^{(4)}) + \Delta_1 I(\hat{Y}_r^{(1)}; Y_b^{(1)} | Q) \quad (59)$$

$$(1 - \alpha_b) \Delta_2 I(Y_r^{(2)}; \hat{Y}_r^{(2)} | Q) + \Delta_1 I(Y_r^{(1)}; \hat{Y}_r^{(1)} | X_a^{(1)}, Q) < \Delta_4 I(X_r^{(4)}; Y_a^{(4)}) + \Delta_2 I(\hat{Y}_r^{(2)}; Y_a^{(2)} | Q) \quad (60)$$

where  $0 < \alpha_a, \alpha_b < 1$  over all joint distributions,

$$p(q, x_a, x_b, x_r, y_a, y_b, y_r, \hat{y}_r) = p(q)p^{(1)}(x_a, y_b, y_r, \hat{y}_r|q)p^{(2)}(x_b, y_a, y_r, \hat{y}_r|q)p^{(3)}(u_a, u_b, x_r, y_a, y_b|q)p^{(4)}(x_r, y_a, y_b) \quad (61)$$

where

$$p^{(1)}(x_a, y_b, y_r, \hat{y}_r|q) = p^{(1)}(x_a|q)p^{(1)}(y_b, y_r|x_a)p^{(1)}(\hat{y}_r|y_r, q) \quad (62)$$

$$p^{(2)}(x_b, y_a, y_r, \hat{y}_r|q) = p^{(2)}(x_b|q)p^{(2)}(y_a, y_r|x_b)p^{(2)}(\hat{y}_r|y_r, q) \quad (63)$$

$$p^{(3)}(u_a, u_b, x_r, y_a, y_b|q) = p^{(3)}(u_a, u_b|q)p^{(3)}(x_r|u_a, u_b, q)p^{(3)}(y_a, y_b|x_r) \quad (64)$$

$$p^{(4)}(x_r, y_a, y_b) = p^{(4)}(x_r)p^{(4)}(y_a, y_b|x_r) \quad (65)$$

with  $|\mathcal{Q}| \leq 13$  over the alphabet  $\mathcal{X}_a \times \mathcal{X}_b \times \mathcal{X}_r^2 \times \mathcal{Y}_a^3 \times \mathcal{Y}_b^3 \times \mathcal{Y}_r^2 \times \mathcal{Y}_r^2$ .  $\square$

*Remark 13:* If side information is very limited, then with  $\Delta_4 \rightarrow 0$ ,  $\alpha_a, \alpha_b \rightarrow 1$  the relay phase acts as a classical broadcast channel. At the opposite extreme, the side information cancels out all interference, i.e.,  $I(Y_r^{(1)}; \hat{Y}_r^{(1)}|X_a^{(1)}) = I(Y_r^{(2)}; \hat{Y}_r^{(2)}|X_b^{(2)}) = 0$ . Then we set  $\Delta_3 \rightarrow 0$ ,  $\alpha_a, \alpha_b \rightarrow 0$ .<sup>4</sup>

*Proof: Random code generation:* For simplicity of exposition, we take  $|\mathcal{Q}| = 1$ .

1) Phase 1: Generate random  $(n \cdot \Delta_{1,n})$ -length sequences

- $\mathbf{x}_a^{(1)}(w_a)$  i.i.d. with  $p^{(1)}(x_a)$ ,  $w_a \in \mathcal{S}_a = \{0, 1, \dots, \lfloor 2^{nR_a} \rfloor - 1\}$
- $\hat{\mathbf{y}}_r^{(1)}(w_{a0})$  i.i.d. with  $p^{(1)}(\hat{y}_r) = \sum_{y_r} p^{(1)}(y_r)p^{(1)}(\hat{y}_r|y_r)$ ,  $w_{a0} \in \{0, 1, \dots, \lfloor 2^{nR_{a0}} \rfloor - 1\} := \mathcal{S}_{a0}$

and generate a partition of  $\mathcal{S}_{a0}$  randomly by independently assigning every index  $w_{a0} \in \mathcal{S}_{a0}$  to a set  $\mathcal{S}_{a0,i}$ , with a uniform distribution over the indices  $i \in \{0, \dots, \lfloor 2^{nR_{a1}} \rfloor - 1\} := \mathcal{S}_{a1}$ . We denote by  $s_{a0}(w_{a0})$  the index  $i$  of  $\mathcal{S}_{a0,i}$  to which  $w_{a0}$  belongs.

2) Phase 2: Generate random  $(n \cdot \Delta_{2,n})$ -length sequences

- $\mathbf{x}_b^{(2)}(w_b)$  i.i.d. with  $p^{(2)}(x_b)$ ,  $w_b \in \mathcal{S}_b = \{0, 1, \dots, \lfloor 2^{nR_b} \rfloor - 1\}$
- $\hat{\mathbf{y}}_r^{(2)}(w_{b0})$  i.i.d. with  $p^{(2)}(\hat{y}_r) = \sum_{y_r} p^{(2)}(y_r)p^{(2)}(\hat{y}_r|y_r)$ ,  $w_{b0} \in \{0, 1, \dots, \lfloor 2^{nR_{b0}} \rfloor - 1\} := \mathcal{S}_{b0}$

and generate a partition of  $\mathcal{S}_{b0}$  randomly by independently assigning every index  $w_{b0} \in \mathcal{S}_{b0}$  to a set  $\mathcal{S}_{b0,i}$ , with a uniform distribution over the indices  $i \in \{0, \dots, \lfloor 2^{nR_{b1}} \rfloor - 1\} := \mathcal{S}_{b1}$ . We denote by  $s_{b0}(w_{b0})$  the index  $i$  of  $\mathcal{S}_{b0,i}$  to which  $w_{b0}$  belongs.

3) Phase 3: Generate random  $(n \cdot \Delta_{3,n})$ -length sequences

- $\mathbf{u}_a^{(3)}(w_{a2})$  i.i.d. with  $p^{(3)}(u_a)$ ,  $w_{a2} \in \{0, 1, \dots, \lfloor 2^{nR_{a2}} \rfloor - 1\} := \mathcal{S}_{a2}$

<sup>4</sup> This choice of  $\Delta_3, \alpha_a, \alpha_b$  is on the boundary of the closure of the achievable rate region.

- $\mathbf{u}_b^{(3)}(w_{b2})$  i.i.d with  $p^{(3)}(u_b)$ ,  $w_{b2} \in \{0, 1, \dots, \lfloor 2^{nR_{b2}} \rfloor - 1\} := \mathcal{S}_{b2}$

and define bin  $B_j := \{w_{a2} | w_{a2} \in [(j-1) \cdot \lfloor 2^{n(R_{a2}-R_{a1})} \rfloor + 1, j \cdot \lfloor 2^{n(R_{a2}-R_{a1})} \rfloor]\}$  for  $j \in \mathcal{S}_{a1}$ .

Likewise,  $C_k := \{w_{b2} | w_{b2} \in [(k-1) \cdot \lfloor 2^{n(R_{b2}-R_{b1})} \rfloor + 1, k \cdot \lfloor 2^{n(R_{b2}-R_{b1})} \rfloor]\}$  for  $k \in \mathcal{S}_{b1}$ .

4) Phase 4: Generate random  $(n \cdot \Delta_{4,n})$ -length sequences

- $\mathbf{x}_r^{(4)}(w_{a0}, w_{b0})$  i.i.d with  $p^{(4)}(x_r)$ ,  $w_{a0} \in \mathcal{S}_{a0}$  and  $w_{b0} \in \mathcal{S}_{b0}$ .

*Encoding:* During phase 1 (resp. phase 2), the encoder of node a (resp. b) sends the codeword  $\mathbf{x}_a^{(1)}(w_a)$  (resp.  $\mathbf{x}_b^{(2)}(w_b)$ ). At the end of phase 1, relay r compresses the received signal  $\mathbf{y}_r^{(1)}$  into the message  $w_{a0}$  if there exists a  $w_{a0}$  such that  $(\mathbf{y}_r^{(1)}, \hat{\mathbf{y}}_r^{(1)}(w_{a0})) \in A^{(1)}(Y_r \hat{Y}_r)$ . Similarly, r compresses  $\mathbf{y}_r^{(2)}$  into the message  $w_{b0}$  at the end of phase 2. There exist such  $w_{a0}$  and  $w_{b0}$  with high probability if

$$R_{a0} = \Delta_{1,n} I(Y_r^{(1)}; \hat{Y}_r^{(1)}) + \epsilon \quad (66)$$

$$R_{b0} = \Delta_{2,n} I(Y_r^{(2)}; \hat{Y}_r^{(2)}) + \epsilon \quad (67)$$

and  $n$  is sufficiently large. Also we choose  $R_{a1}$ ,  $R_{b1}$ ,  $R_{a2}$  and  $R_{b2}$  as:

$$R_{a1} = \alpha_a R_{a0} = \alpha_a (\Delta_{1,n} I(Y_r^{(1)}; \hat{Y}_r^{(1)}) + \epsilon) \quad (68)$$

$$R_{b1} = \alpha_b R_{b0} = \alpha_b (\Delta_{2,n} I(Y_r^{(2)}; \hat{Y}_r^{(2)}) + \epsilon) \quad (69)$$

and

$$R_{a1} \leq R_{a2} = \Delta_{3,n} I(U_a^{(3)}; Y_b^{(3)}) - 4\epsilon \quad (70)$$

$$R_{b1} \leq R_{b2} = \Delta_{3,n} I(U_b^{(3)}; Y_a^{(3)}) - 4\epsilon. \quad (71)$$

From the code constructions of  $w_{a2}$  and  $w_{b2}$ ,  $R_{a1}$  and  $R_{b1}$  have to be less than  $R_{a2}$  and  $R_{b2}$ , respectively.

Then the relay constructs  $w_{a1} = s_{a0}(w_{a0})$  and  $w_{b1} = s_{b0}(w_{b0})$ . To choose  $w_{a2}$  and  $w_{b2}$ , the relay first selects the bins  $B_{w_{a1}}$  and  $C_{w_{b1}}$  and then it searches for a pair  $(w_{a2}, w_{b2}) \in B_{w_{a1}} \times C_{w_{b1}}$  such that  $(\mathbf{u}_a^{(3)}(w_{a2}), \mathbf{u}_b^{(3)}(w_{b2})) \in A^{(3)}(U_a U_b)$ . Such a  $(w_{a2}, w_{b2})$  exists with high probability if

$$R_{a1} + R_{b1} < R_{a2} + R_{b2} - \Delta_{3,n} I(U_a^{(3)}; U_b^{(3)}) - \epsilon' \quad (72)$$

from the Lemma in [6]. The relay then sends  $\mathbf{x}_r^{(3)}$  generated i.i.d. according to  $p^{(3)}(x_r | u_a, u_b)$  with  $\mathbf{u}_a^{(3)}(w_{a2})$  and  $\mathbf{u}_b^{(3)}(w_{b2})$  during phase 3. Finally, the relay sends  $\mathbf{x}_r^{(4)}(w_{a0}, w_{b0})$  during phase 4.

*Decoding:* Node a decodes  $\tilde{w}_{b2}$  after phase 3 using jointly typical decoding. Then a estimates  $\tilde{w}_{b1}$  from the bin index of  $\tilde{w}_{b2}$ . Node a decodes  $\tilde{w}_{b0}$  if there exists a unique  $\tilde{w}_{b0}$  such that  $\tilde{w}_{b0} \in \mathcal{S}_{b0, \tilde{w}_{b1}}$ ,  $(\mathbf{x}_r^{(4)}(\tilde{w}_{a0}, \tilde{w}_{b0}), \mathbf{y}_a^{(4)}) \in A^{(4)}(X_r Y_a)$ ,  $(\mathbf{x}_a^{(1)}(w_a), \hat{\mathbf{y}}_r^{(1)}(\tilde{w}_{a0})) \in A^{(1)}(X_a \hat{Y}_r)$  and  $(\hat{\mathbf{y}}_b^{(2)}(w_{b0}), \mathbf{y}_a^{(2)}) \in A^{(2)}(\hat{Y}_b Y_a)$

. After decoding  $\tilde{w}_{b0}$ , node a decodes  $\tilde{w}_b$  using jointly typical decoding of the sequence  $(\mathbf{x}_b^{(2)}, \hat{\mathbf{y}}_r^{(2)}(\tilde{w}_{b0}), \mathbf{y}_a^{(2)})$ . Similarly, node b decodes  $\tilde{w}_a$ .

*Error analysis:*

$$P[E_{b,a}] \leq P[E_{r,a}^{(3)} \cup E_{r,a}^{(4)} \cup E_{b,a}^{(4)}] \quad (73)$$

$$\leq P[E_{r,a}^{(3)}] + P[E_{r,a}^{(4)} | \bar{E}_{r,a}^{(3)}] + P[E_{b,a}^{(4)} | \bar{E}_{r,a}^{(3)} \cap \bar{E}_{r,a}^{(4)}] \quad (74)$$

We define error events in each phase as follows:

- 1)  $E_{r,a}^{(3)} = E_{r,a}^{(3),1} \cup E_{r,a}^{(3),2} \cup E_{r,a}^{(3),3} \cup E_{r,a}^{(3),4} \cup E_{r,a}^{(3),5}$ .
  - $E_{r,a}^{(3),1}$  : there does not exist a  $w_{a0}$  such that  $(\mathbf{y}_r^{(1)}, \hat{\mathbf{y}}_r^{(1)}(w_{a0})) \in A^{(1)}(Y_r \hat{Y}_r)$ .
  - $E_{r,a}^{(3),2}$  : there does not exist a  $w_{b0}$  such that  $(\mathbf{y}_r^{(2)}, \hat{\mathbf{y}}_r^{(2)}(w_{b0})) \in A^{(2)}(Y_r \hat{Y}_r)$ .
  - $E_{r,a}^{(3),3}$  : there does not exist a pair  $(w_{a2}, w_{b2}) \in B_{s_{a0}(w_{a0})} \times C_{s_{b0}(w_{b0})}$  such that  $(\mathbf{u}_a^{(3)}(w_{a2}), \mathbf{u}_b^{(3)}(w_{b2})) \in A^{(3)}(U_a U_b)$ .
  - $E_{r,a}^{(3),4}$  :  $(\mathbf{u}_b^{(3)}(w_{b2}), \mathbf{y}_a^{(3)}) \notin A^{(3)}(U_b Y_a)$ .
  - $E_{r,a}^{(3),5}$  : there exists  $\tilde{w}_{b2} \neq w_{b2}$  such that  $(\mathbf{u}_b^{(3)}(\tilde{w}_{b2}), \mathbf{y}_a^{(3)}) \in A^{(3)}(U_b Y_a)$ .
- 2)  $E_{r,a}^{(4)} = E_{r,a}^{(4),1} \cup E_{r,a}^{(4),2} \cup E_{r,a}^{(4),3} \cup E_{r,a}^{(4),4}$ .
  - $E_{r,a}^{(4),1}$  : there does not exist a  $w_{a0}$  such that  $(\mathbf{y}_r^{(1)}, \hat{\mathbf{y}}_r^{(1)}(w_{a0})) \in A^{(1)}(Y_r \hat{Y}_r)$ .
  - $E_{r,a}^{(4),2}$  : there does not exist a  $w_{b0}$  such that  $(\mathbf{y}_r^{(2)}, \hat{\mathbf{y}}_r^{(2)}(w_{b0})) \in A^{(2)}(Y_r \hat{Y}_r)$ .
  - $E_{r,a}^{(4),3}$  :  $(\mathbf{x}_r^{(4)}(w_{a0}, w_{b0}), \mathbf{y}_a^{(4)}) \notin A^{(4)}(X_r Y_a)$ .
  - $E_{r,a}^{(4),4}$  : there exists  $(\tilde{w}_{a0}, \tilde{w}_{b0})$  where  $\tilde{w}_{a0} \neq w_{a0}$  and  $\tilde{w}_{b0} \neq w_{b0}$  such that  $(\mathbf{x}_r^{(4)}(\tilde{w}_{a0}, \tilde{w}_{b0}), \mathbf{y}_a^{(4)}) \in A^{(4)}(X_r Y_a)$ ,  $(\mathbf{x}_a^{(1)}(w_a), \hat{\mathbf{y}}_r^{(1)}(\tilde{w}_{a0})) \in A^{(1)}(X_a \hat{Y}_r)$ ,  $(\hat{\mathbf{y}}_r^{(2)}(\tilde{w}_{b0}), \mathbf{y}_a^{(2)}) \in A^{(2)}(\hat{Y}_r Y_a)$ , and  $\tilde{w}_{b0} \in S_{b0, s_{b0}(w_{b0})}$ .
  - $E_{r,a}^{(4),5}$  : there exists  $\tilde{w}_{b0} \neq w_{b0}$  such that  $(\mathbf{x}_r^{(4)}(w_{a0}, \tilde{w}_{b0}), \mathbf{y}_a^{(4)}) \in A^{(4)}(X_r Y_a)$ ,  $(\hat{\mathbf{y}}_r^{(2)}(\tilde{w}_{b0}), \mathbf{y}_a^{(2)}) \in A^{(2)}(\hat{Y}_r Y_a)$ , and  $\tilde{w}_{b0} \in S_{b0, s_{b0}(w_{b0})}$ .
- 3)  $E_{b,a}^{(4)} = E_{b,a}^{(4),1} \cup E_{b,a}^{(4),2}$ .
  - $E_{b,a}^{(4),1}$  :  $(\mathbf{x}_b^{(2)}(w_b), \mathbf{y}_a^{(2)}, \hat{\mathbf{y}}_r^{(2)}(w_{b0})) \notin A^{(2)}(X_b Y_a \hat{Y}_r)$ .
  - $E_{b,a}^{(4),2}$  : there exists  $\tilde{w}_b \neq w_b$  such that  $(\mathbf{x}_b^{(2)}(\tilde{w}_b), \mathbf{y}_a^{(2)}, \hat{\mathbf{y}}_r^{(2)}(w_{b0})) \in A^{(2)}(X_b Y_a \hat{Y}_r)$ .

Then,

$$P[E_{r,a}^{(3)}] \leq P[E_{r,a}^{(3),1}] + P[E_{r,a}^{(3),2}] + P[E_{r,a}^{(3),3}] + P[E_{r,a}^{(3),4}] + P[E_{r,a}^{(3),5}] \quad (75)$$

$$\leq 3\epsilon + P[\bar{D}^{(3)}(\mathbf{u}_b(w_{b2}), \mathbf{y}_a)] + P[\cup_{\tilde{w}_{b2} \neq w_{b2}} D^{(3)}(\mathbf{u}_b(\tilde{w}_{b2}), \mathbf{y}_a)] \quad (76)$$

$$\leq 4\epsilon + 2^{n(R_{b2} - \Delta_{3,n} I(U_b^{(3)}; Y_a^{(3)})) + 3\epsilon} \quad (77)$$

In (76),  $P[E_{r,a}^{(3),1}]$ ,  $P[E_{r,a}^{(3),2}]$  and  $P[E_{r,a}^{(3),3}]$  are less than  $\epsilon$  due to (66), (67) and (72), respectively.

$$P[E_{r,a}^{(4)}|\bar{E}_{r,a}^{(3)}] \leq P[E_{r,a}^{(4),1}|\bar{E}_{r,a}^{(3)}] + P[E_{r,a}^{(4),2}|\bar{E}_{r,a}^{(3)}] + P[E_{r,a}^{(4),3}|\bar{E}_{r,a}^{(3)}] +$$

$$P[E_{r,a}^{(4),4}|\bar{E}_{r,a}^{(3)}] + P[E_{r,a}^{(4),5}|\bar{E}_{r,a}^{(3)}] \quad (78)$$

$$\leq P[\bar{D}^{(4)}(\mathbf{x}_r(w_{a0}, w_{b0}), \mathbf{y}_a)] +$$

$$P\left[\bigcup_{\substack{\tilde{w}_{a0} \neq w_{a0} \\ \tilde{w}_{b0} \neq w_{b0}}} D^{(4)}(\mathbf{x}_r(\tilde{w}_{a0}, \tilde{w}_{b0}), \mathbf{y}_a), D^{(1)}(\mathbf{x}_a(w_a), \hat{\mathbf{y}}_r(\tilde{w}_{a0})),\right.$$

$$D^{(2)}(\hat{\mathbf{y}}_r(\tilde{w}_{b0}), \mathbf{y}_a), s_{b0}(\tilde{w}_{b0}) = w_{b1}] +$$

$$P[\bigcup_{\tilde{w}_{b0} \neq w_{b0}} D^{(4)}(\mathbf{x}_r(w_{a0}, \tilde{w}_{b0}), \mathbf{y}_a), D^{(2)}(\hat{\mathbf{y}}_r(\tilde{w}_{b0}), \mathbf{y}_a), s_{b0}(\tilde{w}_{b0}) = w_{b1}] \quad (79)$$

$$\leq \epsilon + 2^{n(R_{a0} + R_{b0} - \Delta_{4,n}I(X_r^{(4)}; Y_a^{(4)}) - \Delta_{1,n}I(\hat{Y}_r^{(1)}; X_a^{(1)}) - \Delta_{2,n}I(\hat{Y}_r^{(2)}; Y_a^{(2)}) - \alpha_b R_{b0} + \epsilon'')} +$$

$$2^{n(R_{b0} - \Delta_{4,n}I(X_r^{(4)}; Y_a^{(4)}) - \Delta_{2,n}I(\hat{Y}_r^{(2)}; Y_r^{(2)}) - \alpha_b R_{b0} + \epsilon''')} \quad (80)$$

In (79),  $P[E_{r,a}^{(4),1}|\bar{E}_{r,a}^{(3)}]$  and  $P[E_{r,a}^{(4),2}|\bar{E}_{r,a}^{(3)}]$  are zero since  $E_{r,a}^{(4),1}$  and  $E_{r,a}^{(4),2}$  are the same as  $E_{r,a}^{(3),1}$  and  $E_{r,a}^{(3),2}$ , respectively. In (80), the bound for  $R_{b0}$  in the second term is implied by that in the third term since  $R_{a0} - \Delta_{1,n}I(\hat{Y}_r^{(1)}; X_a^{(1)}) = \Delta_{1,n}I(Y_r^{(1)}; \hat{Y}_r^{(1)}|X_a^{(1)}) + \epsilon \geq 0$ .

$$P[E_{b,a}^{(4)}|\bar{E}_{r,a}^{(3)} \cap \bar{E}_{r,a}^{(4)}] \leq P[E_{b,a}^{(4),1}|\bar{E}_{r,a}^{(3)} \cap \bar{E}_{r,a}^{(4)}] + P[E_{b,a}^{(4),2}|\bar{E}_{r,a}^{(3)} \cap \bar{E}_{r,a}^{(4)}] \quad (81)$$

$$= P[\bar{D}^{(2)}(\mathbf{x}_b(w_b), \mathbf{y}_b, \hat{\mathbf{y}}_r(w_{b0}))] + P[\bigcup_{\tilde{w}_b \neq w_b} D^{(2)}(\mathbf{x}_b(\tilde{w}_b), \mathbf{y}_b, \hat{\mathbf{y}}_r(w_{b0}))] \quad (82)$$

$$\leq \epsilon + 2^{n(R_b - \Delta_{2,n}I(X_b^{(2)}; \hat{Y}_r^{(2)}, Y_a^{(2)}) + 3\epsilon)} \quad (83)$$

Since  $\epsilon > 0$  is arbitrary, a proper choice of  $\alpha_b$ , the conditions of Theorem 12, (71), and the AEP property guarantee that the right hand sides of (77), (80) and (83) corresponding to (71), (60) and (55) vanish as  $n \rightarrow \infty$ . Similarly,  $P[E_{a,b}] \rightarrow 0$  as  $n \rightarrow \infty$ . By the Carathéodory theorem in [9], it is sufficient to restrict  $|\mathcal{Q}| \leq 13$ . ■

When the  $h_a$  and  $h_b$  links are of different strength, a scheme in which one link uses CF and the other uses DF may provide a larger rate region than if both links use CF. In the next theorem, we provide a rate region for a TDBC scenario in which the forward link uses DF and the reverse link uses CF.

*Theorem 14:* An achievable rate region for the half-duplex bi-directional relay channel with a mixed TDBC protocol, where the  $a \rightarrow r \rightarrow b$  link uses decode and forward and the  $b \rightarrow r \rightarrow a$  link uses compress and forward, is the closure of the set of all points  $(R_a, R_b)$  satisfying

$$R_a < \min \left\{ \Delta_1 I(X_a^{(1)}; Y_r^{(1)}), \Delta_1 I(X_a^{(1)}; Y_b^{(1)}) + \Delta_3 I(U_r^{(3)}; Y_b^{(3)}|Q) - \Delta_3 I(U_r^{(3)}; U_b^{(3)}|Q) \right\} \quad (84)$$

$$R_b < \Delta_2 I(X_b^{(2)}; \hat{Y}_r^{(2)}, Y_a^{(2)}|Q) \quad (85)$$

subject to

$$\Delta_2 I(Y_r^{(2)}; \hat{Y}_r^{(2)} | Y_a^{(2)}, Q) < \min\{\Delta_3 I(U_r^{(3)}, U_b^{(3)}; Y_a^{(3)} | Q), \Delta_3 I(U_b^{(3)}; U_r^{(3)}, Y_a^{(3)} | Q)\} \quad (86)$$

over all joint distributions,

$$p(q, x_a, x_b, x_r, u_b, u_r, y_a, y_b, y_r, \hat{y}_r) = p(q) p^{(1)}(x_a, y_b, y_r) p^{(2)}(x_b, y_a, y_r, \hat{y}_r | q) p^{(3)}(u_b, u_r, x_r, y_a, y_b | q) \quad (87)$$

where

$$p^{(1)}(x_a, y_b, y_r) = p^{(1)}(x_a) p^{(1)}(y_b, y_r | x_a) \quad (88)$$

$$p^{(2)}(x_b, y_a, y_r, \hat{y}_r | q) = p^{(2)}(x_b | q) p^{(2)}(y_a, y_r | x_b) p^{(2)}(\hat{y}_r | y_r, q) \quad (89)$$

$$p^{(3)}(u_a, u_b, u_r, x_r, y_a, y_b | q) = p^{(3)}(u_b, u_r | q) p^{(3)}(x_r | u_b, u_r, q) p^{(3)}(y_a, y_b | x_r) \quad (90)$$

with  $|Q| \leq 6$  over the alphabet  $\mathcal{X}_a \times \mathcal{X}_b \times \mathcal{X}_r \times \mathcal{U}_b \times \mathcal{U}_r \times \mathcal{Y}_a^2 \times \mathcal{Y}_b^2 \times \mathcal{Y}_r^2 \times \hat{\mathcal{Y}}_r$ .  $\square$

*Remark 15:* We use random binning and a Gel'fand-Pinsker coding scheme in Theorem 14. The detailed proof is provided in Appendix I.

In contrast to the MABC protocols, in the TDBC protocols, the mixed TDBC protocol is not outer bounded by the DF TDBC protocol. In the TDBC protocol, each terminal node obtains side information, used during decoding, when the opposite node transmits (phase 1 for b and phase 2 for a). The data rate as well as the phase durations  $\Delta_1$  and  $\Delta_2$  of the TDBC protocol may vary, and we note that one can easily find cases in which the mixed TDBC protocol outperforms the DF TDBC protocol. For example, suppose the channel of the link  $r \leftrightarrow a$  is good enough such that  $I(X_r^{(3)}; Y_a^{(3)}) \geq \min\{I(U_r^{(3)}, U_b^{(3)}; Y_a^{(3)}), I(U_b^{(3)}; U_r^{(3)}, Y_a^{(3)})\} \gg 0$  and the link mutual informations between  $b \leftrightarrow r$  and  $b \leftrightarrow a$  are the same such that  $I(X_b^{(2)}; Y_r^{(2)}) = I(X_b^{(2)}; Y_a^{(2)})$ . Furthermore, take  $\Delta_1 = \epsilon_1$ ,  $\Delta_3 = \epsilon_2$ , where  $\epsilon_1$  and  $\epsilon_2$  are small positive numbers. We assume the input distributions are fixed, i.e.  $|Q| = 1$ . Then, in the DF TDBC protocol, from Theorem 4, an achievable rate region is given by:

$$R_a < \epsilon' \quad (91)$$

$$R_b < (1 - \epsilon_1 - \epsilon_2) I(X_b^{(2)}; Y_a^{(2)}) \approx I(X_b^{(2)}; Y_a^{(2)}), \quad (92)$$

for some  $\epsilon'$ . In the mixed TDBC protocol, we find a choice for  $\min\{I(U_r^{(3)}, U_b^{(3)}; Y_a^{(3)}), I(U_b^{(3)}; U_r^{(3)}, Y_a^{(3)})\}$  which satisfies (86) for the same  $\Delta_i$ 's (this is possible as the  $r \leftrightarrow a$  channel is very strong). Then we obtain the following achievable rate region from Theorem 14 :

$$R_a < \epsilon'' \quad (93)$$

$$R_b < (1 - \epsilon_1 - \epsilon_2) I(X_b^{(2)}; Y_a^{(2)}, \hat{Y}_r^{(2)}) \approx I(X_b^{(2)}; Y_a^{(2)}) + I(X_b^{(2)}; \hat{Y}_r^{(2)} | Y_a^{(2)}), \quad (94)$$

for some  $\epsilon''$ . By properly choosing  $\epsilon_1$  and  $\epsilon_2$ , the regions bounded by (91), (92), (93) and (94) demonstrate the case in which a larger rate region is achieved through the mixed TDBC protocol.

#### IV. GAUSSIAN CASE

We now assume all links in the bi-directional relay channel are subject to independent, identically distributed white Gaussian noise. The commonly considered Gaussian channel will allow us to visually compare different achievable rate regions for the bi-directional relaying channel. Definitions of codes, rate, and achievability in the memoryless Gaussian channels are analogous to those of the discrete memoryless channels.

We apply the previous results to the Gaussian channel. Since strong typicality does not apply to continuous random variables, the achievable rate regions from the theorems in the previous section do not directly apply to continuous domains. However, for the Gaussian input distributions and additive Gaussian noise which we will assume in the following, the Markov lemma of [17], which generalizes the Markov lemma to the continuous domains, ensures that the achievable rate regions in the previous section hold in the Gaussian case. We use Gaussian input distributions since we assume the average power constraint.

The corresponding Gaussian channel model is:

$$Y_a[m] = h_{ra}X_r[m] + h_{ba}X_b[m] + Z_a[m] \quad (95)$$

$$Y_b[m] = h_{rb}X_r[m] + h_{ab}X_a[m] + Z_b[m] \quad (96)$$

$$Y_r[m] = h_{ar}X_a[m] + h_{br}X_b[m] + Z_r[m] \quad (97)$$

where  $X_a[m]$ ,  $X_b[m]$  and  $X_r[m]$  follow the input distributions  $X_a^{(\ell)} \sim \mathcal{CN}(0, P_a)$ ,  $X_b^{(\ell)} \sim \mathcal{CN}(0, P_b)$  and  $X_r^{(\ell)} \sim \mathcal{CN}(0, P_r)$  respectively during transmitting, where  $m \in [n \sum_{j=0}^{\ell-1} \Delta_{j,n} + 1, n \sum_{j=0}^{\ell} \Delta_{j,n}]$  and  $\mathcal{CN}(\mu, \sigma^2)$  denotes a complex Gaussian random variable with mean  $\mu$  and variance  $\sigma^2$ , and  $\ell$  corresponds to the appropriate phase. If node  $i$  is in transmitting mode, the transmit power is bounded by  $P_i$ , i.e.,  $E[X_i^2] \leq P_i$ . If node  $i$  is in receiving mode, the input symbol does not exist in the above mathematical channel model. For example, in the first phase of the TDBC protocol, the corresponding channel model is :

$$Y_b[m] = h_{ab}X_a[m] + Z_b[m] \quad (98)$$

$$Y_r[m] = h_{ar}X_a[m] + Z_r[m]. \quad (99)$$



TABLE IV  
INPUT AND OUTPUT DISTRIBUTIONS

Dist. of	during TX	during RX
$X_a^{(\ell)}$	$\mathcal{CN}(0, P_a)$	N/A
$X_b^{(\ell)}$	$\mathcal{CN}(0, P_b)$	N/A
$X_r^{(\ell)}$	$\mathcal{CN}(0, P_r)$	N/A
$Z_a^{(\ell)} = Z_b^{(\ell)} = Z_r^{(\ell)}$	N/A	$\mathcal{CN}(0, 1)$
$\hat{Y}_r^{(\ell)}$	N/A	$\mathcal{CN}(0, P_{\hat{y}}^{(\ell)})$
$Z_{\hat{r}}^{(\ell)}$	N/A	$\mathcal{CN}\left(0, P_{\hat{y}}^{(\ell)} - \frac{(\sigma_y^{(\ell)})^2}{P_{\hat{y}}^{(\ell)}}\right)$

In the above  $h_{ij}$  is the effective channel gain between transmitter  $i$  and receiver  $j$ , which is modeled as a complex number. We assume that the channel is reciprocal such that  $h_{ij} = h_{ji}$  and each node is fully aware of  $h_{ar}$ ,  $h_{br}$  and  $h_{ab}$  (i.e. we have full CSI). The noise at all receivers  $Z_a, Z_b, Z_r$  is of unit power, additive, white Gaussian, complex and circularly symmetric. For convenience of analysis, we also define the function  $C(x) := \log_2(1 + x)$ .

For the analysis of the Compress and Forward scheme, we assume  $\hat{Y}_r^{(\ell)}$  are zero mean Gaussians and define  $P_y^{(\ell)} := E[(Y_r^{(\ell)})^2]$ ,  $P_{\hat{y}}^{(\ell)} := E[(\hat{Y}_r^{(\ell)})^2]$  and  $\sigma_y^{(\ell)} := E[\hat{Y}_r^{(\ell)} Y_r^{(\ell)}]$ . Then the relation between the received  $Y_r[m]$  and the compressed  $\hat{Y}_r[m]$  are given by the following equivalent channel model:

$$\hat{Y}_r[m] = h_{r\hat{r}}[m]Y_r[m] + Z_{\hat{r}}[m] \quad (100)$$

where  $Y_r[m]$ ,  $\hat{Y}_r[m]$  and  $Z_{\hat{r}}[m]$  follow the distributions  $Y_r^{(\ell)} \sim \mathcal{CN}(0, P_y^{(\ell)})$ ,  $\hat{Y}_r^{(\ell)} \sim \mathcal{CN}(0, P_{\hat{y}}^{(\ell)})$  and  $Z_{\hat{r}}^{(\ell)} \sim \mathcal{CN}(0, P_{\hat{y}}^{(\ell)} - \frac{(\sigma_y^{(\ell)})^2}{P_{\hat{y}}^{(\ell)}})$  and  $h_{r\hat{r}}[m] = \frac{\sigma_y^{(\ell)}}{P_{\hat{y}}^{(\ell)}}$ , where  $m \in [n \sum_{j=0}^{\ell-1} \Delta_{j,n} + 1, n \sum_{j=0}^{\ell} \Delta_{j,n}]$ . We note that in the following,  $P_{\hat{y}}^{(\ell)}$  and  $\sigma_y^{(\ell)}$  are unknown variables corresponding to the quantization which we numerically optimize.

We consider four different relaying schemes (i.e. ways in which the relay processes and forwards the received signal) for each MABC and TDBC bi-directional protocol: *Amplify and Forward (AF)*, *Decode and Forward (DF)*, *Compress and Forward (CF)*, and *Mixed Forward (Mixed)*. In addition to achievable rate regions, we apply outer bounds of the MABC and TDBC protocols to the Gaussian channel.

#### A. Amplify and Forward

In the amplify and forward scheme, all phase durations are equal, since relaying is performed on a symbol by symbol basis. Therefore,  $\Delta_1 = \Delta_2 = \frac{1}{2}$  for the MABC protocol and  $\Delta_1 = \Delta_2 = \Delta_3 = \frac{1}{3}$

for the TDBD protocol. Furthermore, relay  $r$  scales the received symbol  $y_r$  by  $\sqrt{\frac{P_r}{P_y}}$  to meet the transmit power constraint of  $P_r$ . The following are achievable rate regions for the amplify and forward relaying:

- MABC Protocol

$$R_a < \frac{1}{2}C \left( \frac{|h_{ar}|^2 |h_{br}|^2 P_a P_r}{|h_{ar}|^2 P_a + |h_{br}|^2 P_b + |h_{br}|^2 P_r + 1} \right) \quad (101)$$

$$R_b < \frac{1}{2}C \left( \frac{|h_{ar}|^2 |h_{br}|^2 P_b P_r}{|h_{ar}|^2 P_a + |h_{br}|^2 P_b + |h_{ar}|^2 P_r + 1} \right) \quad (102)$$

- TDBC Protocol

$$R_a < \frac{1}{3}C \left( |h_{ab}|^2 P_a + \frac{|h_{ar}|^2 |h_{br}|^2 P_a P_r}{|h_{ar}|^2 P_a + |h_{br}|^2 P_b + 2|h_{br}|^2 P_r + 2} \right) \quad (103)$$

$$R_b < \frac{1}{3}C \left( |h_{ab}|^2 P_b + \frac{|h_{ar}|^2 |h_{br}|^2 P_b P_r}{|h_{ar}|^2 P_a + |h_{br}|^2 P_b + 2|h_{ar}|^2 P_r + 2} \right) \quad (104)$$

### B. Decode and Forward

Applying Theorems 3 and 4 to the Gaussian case, we obtain the following achievable rate regions:

- MABC Protocol

$$R_a < \min\{\Delta_1 C(|h_{ar}|^2 P_a), \Delta_2 C(|h_{br}|^2 P_r)\} \quad (105)$$

$$R_b < \min\{\Delta_1 C(|h_{br}|^2 P_b), \Delta_2 C(|h_{ar}|^2 P_r)\} \quad (106)$$

$$R_a + R_b < \Delta_1 C(|h_{ar}|^2 P_a + |h_{br}|^2 P_b) \quad (107)$$

- TDBC Protocol

$$R_a < \min\{\Delta_1 C(|h_{ar}|^2 P_a), \Delta_1 C(|h_{ab}|^2 P_a) + \Delta_3 C(|h_{br}|^2 P_r)\} \quad (108)$$

$$R_b < \min\{\Delta_2 C(|h_{br}|^2 P_b), \Delta_2 C(|h_{ab}|^2 P_b) + \Delta_3 C(|h_{ar}|^2 P_r)\} \quad (109)$$

When obtaining the regions numerically, we optimize  $\Delta_\ell$ 's for the given channel mutual informations to maximize the achievable rate regions.

### C. Compress and Forward

Applying Theorem 5 and 12 to the Gaussian case, we obtain the following achievable rate regions:

- MABC Protocol

$$R_a < \Delta_1 C \left( \frac{(\sigma_y^{(1)})^2 |h_{ar}|^2 P_a}{P_{\hat{y}}^{(1)} (P_y^{(1)})^2 - (\sigma_y^{(1)})^2 (P_y^{(1)} - 1)} \right) \quad (110)$$

$$R_b < \Delta_1 C \left( \frac{(\sigma_y^{(1)})^2 |h_{br}|^2 P_b}{P_{\hat{y}}^{(1)} (P_y^{(1)})^2 - (\sigma_y^{(1)})^2 (P_y^{(1)} - 1)} \right) \quad (111)$$

where,

$$\Delta_1 = \min \left\{ \frac{C(|h_{br}|^2 P_r)}{C \left( \frac{(\sigma_y^{(1)})^2 (|h_{ar}|^2 P_a + 1)}{P_y^{(1)} (P_y^{(1)})^2 - (\sigma_y^{(1)})^2 P_y^{(1)}} \right) + C(|h_{br}|^2 P_r)}, \frac{C(|h_{ar}|^2 P_r)}{C \left( \frac{(\sigma_y^{(1)})^2 (|h_{br}|^2 P_b + 1)}{P_y^{(1)} (P_y^{(1)})^2 - (\sigma_y^{(1)})^2 P_y^{(1)}} \right) + C(|h_{ar}|^2 P_r)} \right\} \quad (112)$$

$$P_y^{(1)} = |h_{ar}|^2 P_a + |h_{br}|^2 P_b + 1 \quad (113)$$

- TDBC Protocol

One can show that (56) – (58) Marton's bound is equivalent to the capacity region of the Gaussian broadcast channel with Costa's setup as follows: let  $|h_{ra}| > |h_{rb}|$  and we set

$$\text{In phase 3} \begin{cases} U_b[m] = V_r[m] + \alpha U_a[m] \\ Y_a[m] = h_{ra}(V_r[m] + U_a[m]) + Z_a[m] \\ Y_b[m] = h_{rb}(V_r[m] + U_a[m]) + Z_b[m] \end{cases} \quad (114)$$

where  $V_r[m]$  and  $U_a[m]$  follow the distributions  $V_r^{(3)} \sim \mathcal{CN}(0, \beta P_r)$ ,  $U_a^{(3)} \sim \mathcal{CN}(0, (1 - \beta) P_r)$  respectively during phase 3,  $m \in [n(\Delta_{1,n} + \Delta_{2,n}) + 1, n]$ , where  $(0 \leq \beta \leq 1)$  and  $E[V_r^{(3)} U_b^{(3)}] = 0$ , i.e.,  $V_r^{(3)}$ ,  $U_a^{(3)}$  are independent. Also we take  $\alpha = \frac{|h_{ra}|^2 \beta P_r}{|h_{ra}|^2 \beta P_r + 1}$ . Then

$$\begin{cases} I(U_a^{(3)}; Y_b^{(3)}) = C \left( \frac{|h_{rb}|^2 (1 - \beta) P_r}{|h_{rb}|^2 \beta P_r + 1} \right) \\ I(U_b^{(3)}; Y_a^{(3)}) - I(U_a^{(3)}; U_b^{(3)}) = C(|h_{ra}|^2 \beta P_r) \end{cases} \quad (115)$$

Similarly, we obtain the bounds in the case  $|h_{ra}| \leq |h_{rb}|$ . These are the same as the capacity region of the Gaussian broadcast channel ((15.11) and (15.12) in [4]).

$$R_a < \Delta_1 C \left( |h_{ab}|^2 P_a + \frac{(\sigma_y^{(1)})^2 |h_{ar}|^2 P_a}{P_y^{(1)} (P_y^{(1)})^2 - (\sigma_y^{(1)})^2 (P_y^{(1)} - 1)} \right) \quad (116)$$

$$R_b < \Delta_2 C \left( |h_{ab}|^2 P_b + \frac{(\sigma_y^{(2)})^2 |h_{br}|^2 P_b}{P_y^{(2)} (P_y^{(2)})^2 - (\sigma_y^{(2)})^2 (P_y^{(2)} - 1)} \right) \quad (117)$$

where,

$$\text{If } |h_{ra}| < |h_{rb}| : \alpha_a \Delta_1 C \left( \frac{(\sigma_y^{(1)})^2}{P_{\hat{y}}^{(1)} P_y^{(1)} - (\sigma_y^{(1)})^2} \right) < \Delta_3 C (\beta |h_{rb}|^2 P_r) \quad (118)$$

$$\alpha_b \Delta_2 C \left( \frac{(\sigma_y^{(2)})^2}{P_{\hat{y}}^{(2)} P_y^{(2)} - (\sigma_y^{(2)})^2} \right) < \Delta_3 C \left( \frac{(1-\beta) |h_{ra}|^2 P_r}{\beta |h_{ra}|^2 P_r + 1} \right), \quad (119)$$

$$\text{otherwise} : \alpha_a \Delta_1 C \left( \frac{(\sigma_y^{(1)})^2}{P_{\hat{y}}^{(1)} P_y^{(1)} - (\sigma_y^{(1)})^2} \right) < \Delta_3 C \left( \frac{(1-\beta) |h_{rb}|^2 P_r}{\beta |h_{rb}|^2 P_r + 1} \right) \quad (120)$$

$$\alpha_b \Delta_2 C \left( \frac{(\sigma_y^{(2)})^2}{P_{\hat{y}}^{(2)} P_y^{(2)} - (\sigma_y^{(2)})^2} \right) < \Delta_3 C (\beta |h_{ra}|^2 P_r), \quad (121)$$

and

$$\begin{aligned} (1 - \alpha_a) \Delta_1 C \left( \frac{(\sigma_y^{(1)})^2}{P_{\hat{y}}^{(1)} P_y^{(1)} - (\sigma_y^{(1)})^2} \right) + \Delta_2 C \left( \frac{(\sigma_y^{(2)})^2}{P_{\hat{y}}^{(2)} (P_y^{(2)})^2 - (\sigma_y^{(2)})^2 P_y^{(2)}} \right) \\ < \Delta_3 C (|h_{br}|^2 P_r) + \Delta_1 C \left( \frac{(\sigma_y^{(1)})^2 |h_{ab}|^2 |h_{ra}|^2 P_a}{(P_y^{(1)})^2 P_{\hat{y}}^{(1)} (|h_{ab}|^2 P_a + 1) - (\sigma_y^{(1)})^2 |h_{ab}|^2 |h_{ra}|^2 P_a} \right) \end{aligned} \quad (122)$$

$$\begin{aligned} (1 - \alpha_b) \Delta_2 C \left( \frac{(\sigma_y^{(2)})^2}{P_{\hat{y}}^{(2)} P_y^{(2)} - (\sigma_y^{(2)})^2} \right) + \Delta_1 C \left( \frac{(\sigma_y^{(1)})^2}{P_{\hat{y}}^{(1)} (P_y^{(1)})^2 - (\sigma_y^{(1)})^2 P_y^{(1)}} \right) \\ < \Delta_3 C (|h_{ar}|^2 P_r) + \Delta_2 C \left( \frac{(\sigma_y^{(2)})^2 |h_{ab}|^2 |h_{rb}|^2 P_b}{(P_y^{(2)})^2 P_{\hat{y}}^{(2)} (|h_{ab}|^2 P_b + 1) - (\sigma_y^{(2)})^2 |h_{ab}|^2 |h_{rb}|^2 P_b} \right) \end{aligned} \quad (123)$$

$$P_y^{(1)} = |h_{ar}|^2 P_a + 1 \quad (124)$$

$$P_y^{(2)} = |h_{br}|^2 P_b + 1 \quad (125)$$

$$0 < \alpha_a, \alpha_b, \beta < 1 \quad (126)$$

Again, when numerically obtaining the regions, we optimize  $P_{\hat{y}}^{(\ell)}$ ,  $\sigma_y^{(\ell)}$ ,  $\Delta_\ell$ ,  $\alpha_a$ ,  $\alpha_b$  and  $\beta$  to maximize the region boundary.

#### D. Mixed Forward

Applying Theorem 8 to the Gaussian case with Costa's setup in [3] we have the channel in the relay broadcasting phase in the Mixed MABC protocol as:

$$\text{In phase 2} \begin{cases} U_r[m] = V_r[m] + \alpha U_b[m] \\ Y_a[m] = h_{ra}(V_r[m] + U_b[m]) + Z_a[m] \\ Y_b[m] = h_{rb}(V_r[m] + U_b[m]) + Z_b[m] \end{cases} \quad (127)$$

where  $V_r[m]$  and  $U_b[m]$  follow the distributions  $V_r^{(2)} \sim \mathcal{CN}(0, \beta P_r)$ ,  $U_b^{(2)} \sim \mathcal{CN}(0, (1 - \beta)P_r)$  during phase 2,  $m \in [\Delta_{1,n} \cdot n + 1, n]$ , where  $(0 \leq \beta \leq 1)$ , and  $E[V_r^{(2)} U_b^{(2)}] = 0$ , i.e.,  $V_r^{(2)}$ ,  $U_b^{(2)}$  are independent. Similarly, we construct the channel in the relay broadcasting phase in the Mixed TDBC protocol. Then we obtain the following achievable rate regions, where we numerically optimize  $\alpha$ ,  $\beta$ ,  $P_{\hat{y}}^{(\ell)}$ ,  $\sigma_y^{(\ell)}$  and  $\Delta_\ell$  to maximize their boundary.

- MABC Protocol

$$R_a < \min \left\{ \Delta_1 C \left( \frac{|h_{ar}|^2 P_a}{|h_{br}|^2 P_b + 1} \right), \Delta_2 \log_2 \left( \frac{\beta P_r (|h_{rb}|^2 P_r + 1)}{|h_{rb}|^2 (1 - \alpha)^2 \beta (1 - \beta) P_r^2 + \beta P_r + \alpha^2 (1 - \beta) P_r} \right) \right\} \quad (128)$$

$$R_b < \Delta_1 C \left( \frac{(\sigma_y^{(1)})^2 |h_{br}|^2 P_b}{P_{\hat{y}}^{(1)} (P_y^{(1)})^2 - (\sigma_y^{(1)})^2 (P_y^{(1)} - 1)} \right) \quad (129)$$

where,

$$\Delta_1 C \left( \frac{(\sigma_y^{(1)})^2 (|h_{br}|^2 P_b + 1)}{P_{\hat{y}}^{(1)} (P_y^{(1)})^2 - (\sigma_y^{(1)})^2 P_y^{(1)}} \right) < \min \left\{ \Delta_2 C (|h_{ra}|^2 P_r), \Delta_2 C \left( |h_{ra}|^2 (1 - \alpha)^2 (1 - \beta) P_r + \frac{\alpha^2 (1 - \beta)}{\beta} \right) \right\} \quad (130)$$

$$P_y^{(1)} = |h_{ar}|^2 P_a + |h_{br}|^2 P_b + 1 \quad (131)$$

- TDBC Protocol

$$R_a < \min \left\{ \Delta_1 C (|h_{ar}|^2 P_a), \Delta_1 C (|h_{ab}|^2 P_a) + \Delta_3 \log_2 \left( \frac{\beta P_r (|h_{rb}|^2 P_r + 1)}{|h_{rb}|^2 (1 - \alpha)^2 \beta (1 - \beta) P_r^2 + \beta P_r + \alpha^2 (1 - \beta) P_r} \right) \right\} \quad (132)$$

$$R_b < \Delta_2 C \left( |h_{ab}|^2 P_b + \frac{(\sigma_y^{(2)})^2 |h_{br}|^2 P_b}{P_{\hat{y}}^{(2)} (P_y^{(2)})^2 - (\sigma_y^{(2)})^2 P_y^{(2)} + (\sigma_y^{(2)})^2} \right) \quad (133)$$

where,

$$\Delta_2 C \left( \frac{(\sigma_y^{(2)})^2 (1 - P^*)}{P_{\hat{y}}^{(2)} P_y^{(2)} - (\sigma_y^{(2)})^2} \right) < \min \left\{ \Delta_3 C (|h_{ra}|^2 P_r), \Delta_3 C \left( |h_{ra}|^2 (1 - \alpha)^2 (1 - \beta) P_r + \frac{\alpha^2 (1 - \beta)}{\beta} \right) \right\} \quad (134)$$

$$P_y^{(2)} = |h_{br}|^2 P_b + 1 \quad (135)$$

$$P^* = \frac{|h_{ab}|^2 P_b}{|h_{ab}|^2 P_b + 1} \cdot \frac{|h_{rb}|^2 P_b}{|h_{rb}|^2 P_b + 1} \quad (136)$$

### E. Outer Bound

Applying Theorems 1 and 2 to the Gaussian case, we obtain the following outer bounds. We optimize  $\Delta_\ell$ 's for given channel mutual informations to maximize these outer bounds.

- MABC Protocol

$$R_a \leq \min\{\Delta_1 C(|h_{ar}|^2 P_a), \Delta_2 C(|h_{br}|^2 P_r)\} \quad (137)$$

$$R_b \leq \min\{\Delta_1 C(|h_{br}|^2 P_b), \Delta_2 C(|h_{ar}|^2 P_r)\} \quad (138)$$

- TDBC Protocol

$$R_a \leq \min\{\Delta_1 C(|h_{ar}|^2 P_a + |h_{ab}|^2 P_a), \Delta_1 C(|h_{ab}|^2 P_a) + \Delta_3 C(|h_{br}|^2 P_r)\} \quad (139)$$

$$R_b \leq \min\{\Delta_2 C(|h_{br}|^2 P_b + |h_{ab}|^2 P_b), \Delta_2 C(|h_{ab}|^2 P_b) + \Delta_3 C(|h_{ar}|^2 P_r)\} \quad (140)$$

$$R_a + R_b \leq \Delta_1 C(|h_{ar}|^2 P_a) + \Delta_2 C(|h_{br}|^2 P_b) \quad (141)$$

## V. ACHIEVABLE RATE REGIONS IN THE GAUSSIAN CHANNEL

In order to obtain an intuitive feel for the regions and to illustrate that the regions are not subsets of one another, the bounds described in Section IV are plotted in this section for a number of different channel configurations. We first compare the rate regions obtained by the bi-directional protocols and outer bounds in cases in which the links are symmetric ( $h_{ar} = h_{br} = 1$ ,  $h_{ab} = 0.2$ ) as well as asymmetric ( $h_{ar} = 0.6$ ,  $h_{br} = 20$ ,  $h_{ab} = 0.5$  and  $h_{ar} = 20$ ,  $h_{br} = 0.6$ ,  $h_{ab} = 0.5$ ) for two different transmit SNRs of 0 and 20dB. The protocols considered are:

- 1) AF MABC: Amplify and Forward Multiple Access Broadcast.
- 2) AF TDBC: Amplify and Forward Time Division Broadcast.
- 3) DF MABC: Decode and Forward Multiple Access Broadcast.
- 4) DF TDBC: Decode and Forward Time Division Broadcast.
- 5) CF MABC: Compress and Forward Multiple Access Broadcast.
- 6) CF TDBC: Compress and Forward Time Division Broadcast.
- 7) Mixed MABC: Mixed scheme with Multiple Access Broadcast.
- 8) Mixed TDBC: Mixed scheme with Time Division Broadcast.
- 9) Outer MABC: Outer bound when using MABC protocol.
- 10) Outer TDBC: Outer bound when using TDBC protocol.

We then proceed to examine the maximal sum-rate  $R_a + R_b$  of the ten schemes as a function of the transmit SNR. Finally, we evaluate the maximal sum-rate and maximal constrained sum-rates (that is we

require the rate  $R_a = R_b$  as well as  $R_a = 2R_b$ ) of the schemes as a function of the relay position. The main conclusions to be drawn are that different schemes are optimal under different channel conditions. We provide further discussions in the following subsections.

#### A. Achievable rate region comparisons

We compare the achievable rate regions and outer bounds of the 10 aforementioned protocols for both symmetric and asymmetric source to relay channel gains at transmit SNRs of 0 and 20dB.

1) *Symmetric Case:* In this case  $h_{ar} = h_{br} = 1$  (Figs. 6, 7). In the low SNR regime, the DF MABC protocol dominates the other protocols. The MABC protocol in general outperforms the TDBC protocol as the benefits of side information and reduced interference are relatively small in this regime. The DF scheme outperforms the other schemes since the relatively large amount of noise in the first phase (and the second phase in the TDBC protocol) can be eliminated in the DF scheme, which cannot be done using the other schemes. In contrast, the DF TDBC protocol dominates the other protocols at high SNR since the direct link is strong enough to convey information in this regime.

In the high SNR regime (when  $P_a = P_b = P_r = P$  is sufficiently large), the AF MABC protocol outperforms the DF MABC protocol. From (101), (102), the achievable rate region of the AF MABC protocol is:

$$R_a < \frac{1}{2} \log \left( 1 + \frac{P^2}{3P+1} \right) \approx \frac{1}{2} \log P \quad (142)$$

$$R_b < \frac{1}{2} \log \left( 1 + \frac{P^2}{3P+1} \right) \approx \frac{1}{2} \log P \quad (143)$$

also from (105), (106) and (107), the achievable rate region of the DF MABC protocol is:

$$R_a < \min\{\Delta_1, 1 - \Delta_1\} \cdot \log(1 + P) \approx \min\{\Delta_1, 1 - \Delta_1\} \cdot \log P < \frac{1}{2} \log P \quad (144)$$

$$R_b < \min\{\Delta_1, 1 - \Delta_1\} \cdot \log(1 + P) \approx \min\{\Delta_1, 1 - \Delta_1\} \cdot \log P < \frac{1}{2} \log P \quad (145)$$

$$R_a + R_b < \Delta_1 \log(1 + 2P) \approx \Delta_1 \log P \quad (146)$$

From (146), we can conclude that the achievable rate region of the DF MABC protocol is outer-bounded by the AF MABC protocol.

In the TDBC protocol, the CF scheme does not outperform the DF scheme since the DF uses two parallel channels in phase one and three, while the CF uses one channel in phase one with two receivers. In other words,  $R_a^{DF} < \Delta_1 C(\cdot) + \Delta_3 C(\cdot)$  for the DF as opposed to  $R_a^{CF} < \Delta_1 C(\sum \cdot)$  for the CF scheme. However, under the MABC protocol, the CF scheme outperforms the DF scheme in the high

SNR regime. This is because the interference of the transmission of two terminal nodes affects the DF MABC scheme due to the multiple-access nature but not the CF scheme (as it must not decode the signals).

The achievable rate region of the Mixed TDBC protocol lies between the CF TDBC protocol and the DF TDBC protocol. In the TDBC protocol,  $\max_{R_b} R_a^{MIX} = \max_{R_b} R_a^{DF}$ , where  $R_a^{MIX}$  is the data rate of node a in the Mixed scheme. Here the max is taken over all rates in the achievable rate regions. The  $\max R_a^{MIX}$  is achieved by taking  $\Delta_2 = 0$ . The rate  $R_a^{DF}$  is similarly defined and  $\max R_a^{DF}$  is achieved in an analogous manner. Therefore, the point  $(\max R_a^{DF}, 0)$  lies in both the Mixed scheme and the DF scheme. While in the TDBC scheme a particular rate may be set to 0 by indirectly setting the appropriate interval  $\Delta_i$  to 0, in the MABC protocol this is not possible. In the MABC protocol, even when  $R_a = 0$  or  $R_b = 0$ , the transmit power ( $P_a$  or  $P_b$ ) remains constant in the first phase and does not decrease to 0, acting as the additional noise for the opposite transmission. Therefore,  $\max_{R_b} R_a^{MIX} \leq \max_{R_b} R_a^{DF}$ . This interference seen in MABC protocols is especially pronounced in the high SNR regime, where the gap between the intercept points of the Mixed scheme and the DF scheme is seen to grow as the interference increases (with SNR). We note that this effect is due to our assumption that both transmitters transmit at full power regardless of the rate. If we were to allow for optimization of the transmission power, larger achievable rate regions for the Mixed MABC protocol could result.

In Figs. 6 and 7, the AF scheme is always outer bounded by the CF scheme. We thus expect that when relay r does not know the codebooks of a and b (and hence cannot decode as in the DF scheme), that the CF scheme is a better strategy than the AF scheme.

In the low SNR regime, the achievable rate region of the DF MABC protocol and the outer bound of the MABC protocol are tight, while in the high SNR regime, the achievable rate region of the CF MABC protocol is tight. For the TDBC protocol, there is a very small gap between the achievable rate region of the DF TDBC protocol and the outer bound of the TDBC protocol since interference is not an issue for the TDBC protocol and hence decoding is thus, intuitively, near optimal.

2) *Asymmetric Cases:* In these cases  $h_{ar} = 0.6$ ,  $h_{br} = 20$ ,  $h_{ab} = 0.5$  (Figs. 8, 9) and  $h_{ar} = 20$ ,  $h_{br} = 0.6$ ,  $h_{ab} = 0.5$  (Figs. 10, 11). Note that these two asymmetric cases are different for the mixed forwarding cases, which assume CF in one direction and DF in the other. In the low SNR regime, the CF TDBC and mixed TDBC protocol achieve the best performance in Fig. 8 and Fig. 10, respectively. However, in the high SNR regime, the DF MABC protocol and the DF TDBC protocol yields larger regions than the other protocols. In contrast to the symmetric case, the AF MABC protocol is not outer bounded by the



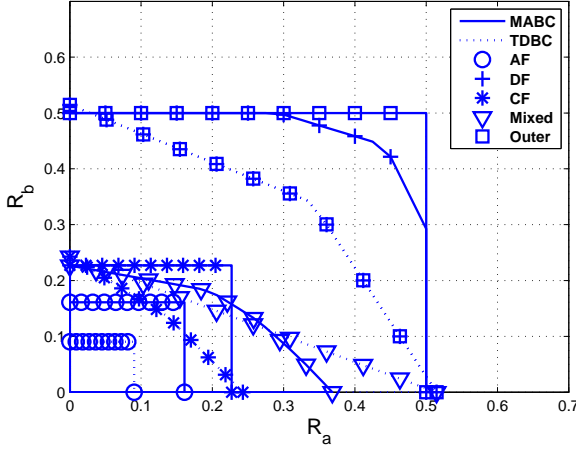


Fig. 6. Comparison of bi-directional regions with  $h_{ar} = h_{br} = 1$ ,  $h_{ab} = 0.2$ ,  $P_a = P_b = P_r = 0$  dB and  $N_a = N_b = N_r = 1$ .

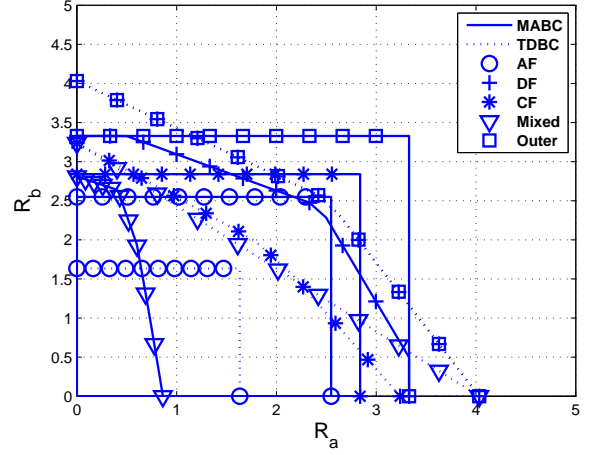


Fig. 7. Comparison of bi-directional regions with  $h_{ar} = h_{br} = 1$ ,  $h_{ab} = 0.2$ ,  $P_a = P_b = P_r = 20$  dB and  $N_a = N_b = N_r = 1$ .

CF MABC protocol.

The mixed forwarding scheme is the only one which has different performance in the two asymmetric cases. In the mixed MABC protocol, if  $h_{br} > h_{ar}$  then the noise seen at relay  $r$  when it decodes  $\tilde{w}_a$  is larger than when  $h_{br} < h_{ar}$ . Therefore, the corresponding achievable rate region is also relatively smaller. In particular, in the high SNR regime (Fig. 9), the achievable rate region for the mixed MABC protocol is only able to achieve, under our input assumptions, rate  $R_a$  close to 0 because of the effective noise from node  $b$  during the first phase. In the mixed TDBC protocol, if  $h_{ar} > h_{br}$ , then we have a larger achievable rate region since the first link is more critical to the performance of the DF scheme. As the SNR increases, the difference between the two asymmetric cases decreases.

### B. Maximum Sum Data Rate

In this subsection we plot the maximum sum-rate  $R_a + R_b$  as a function of the transmit SNR for the symmetric and two asymmetric cases of the previous subsection. As expected, different schemes dominate for different SNR values. The sum-rate is basically proportional to the SNR in dB scale since the sum-rate is roughly the logarithm of the SNR. In Fig. 12 around 12 dB the relative performance of the CF MABC protocol and the DF MABC protocol changes. At lower SNRs, the DF MABC protocol is better, while at higher SNRs, the CF MABC protocol is better. The AF MABC protocol is always worse than the CF MABC protocol in the symmetric case (Fig. 12). In the TDBC protocol, the sum-rate of the mixed TDBC protocol lies between the DF scheme and the CF scheme in Fig. 12.

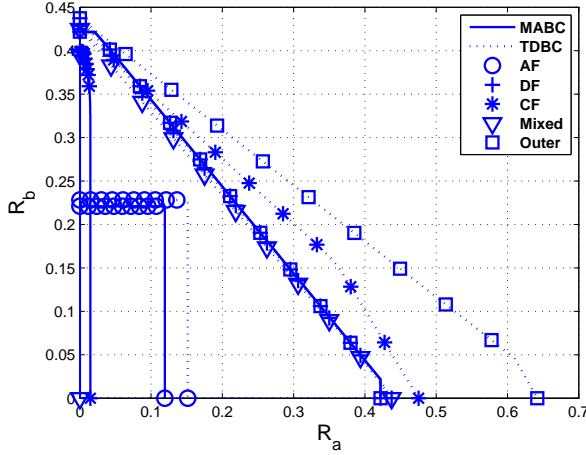


Fig. 8. Comparison of bi-directional regions with  $h_{ar} = 0.6$ ,  $h_{br} = 20$ ,  $h_{ab} = 0.5$ ,  $P_a = P_b = P_r = 0$  dB and  $N_a = N_b = N_r = 1$ .

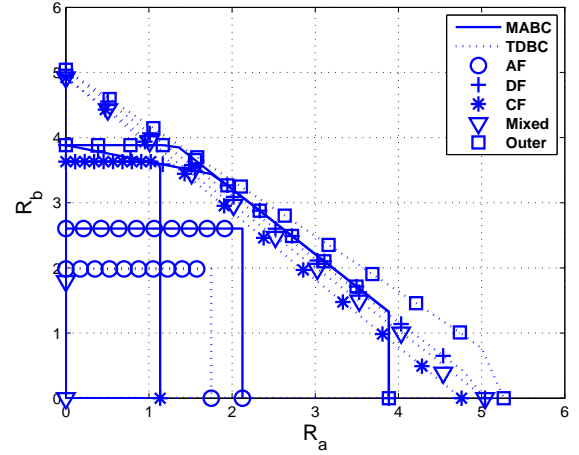


Fig. 9. Comparison of bi-directional regions with  $h_{ar} = 0.6$ ,  $h_{br} = 20$ ,  $h_{ab} = 0.5$ ,  $P_a = P_b = P_r = 20$  dB and  $N_a = N_b = N_r = 1$ .

### C. Relay position

In this subsection we plot the maximum sum-rate  $R_a + R_b$  as a function of the relay position  $d_{ar} = \zeta d_{ab}$  ( $0 < \zeta < 1$ ) when the relay  $r$  is located on the line between  $a$  and  $b$ . Thus,  $d_{br} = (1 - \zeta)d_{ab}$ . We apply  $h_{ab} = 0.2$  and  $P_a = P_b = P_r = 20$  dB and let  $|h_{ij}|^2 = k/d_{ij}^{3.8}$  for  $k$  constant and a path-loss exponent of 3.8. We consider three constraints on the sum-rate in the three Figs. 15, 16 and 17. In the first, the sum-rate is maximized without any additional constraints. For the latter two we consider more realistic scenarios in which the sum-rate is constrained. In many communication systems, uplink and downlink rates are not equal. More specifically, it is not uncommon for the downlink rate to be 2 to 4 times greater than that of the uplink. In Figs. 16 and 17 we plot the maximal sum-rate of the protocols under a  $\sigma = R_a/R_b$  rate ratio restriction. These constrained sum-rates are obtained by optimizing the Gaussian regions of Section IV with the additional constraint  $R_a = \sigma R_b$ .

For the MABC protocol, if the relay location is biased (not midway between nodes  $a$  and  $b$ ), then the DF MABC protocol outperforms the CF MABC protocol and the AF MABC protocol outperforms the CF MABC protocol. This effect is more explicit in the constrained cases. In contrast, for the TDBC protocol, in order of increasing complexity, (and performance), the relaying schemes are AF, CF, Mixed and DF in all cases when ( $0.1 \leq \zeta \leq 0.9$ ). As expected, the sum-rate for the mixed MABC protocol is worse than those of the other protocols.

The sum-rate plot for the mixed protocol is not symmetric since it uses different forwarding schemes

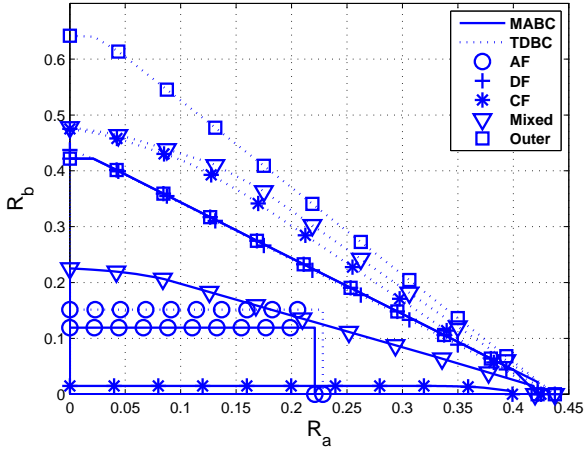


Fig. 10. Comparison of bi-directional regions with  $h_{ar} = 20$ ,  $h_{br} = 0.6$ ,  $h_{ab} = 0.5$ ,  $P_a = P_b = P_r = 0$  dB and  $N_a = N_b = N_r = 1$ .

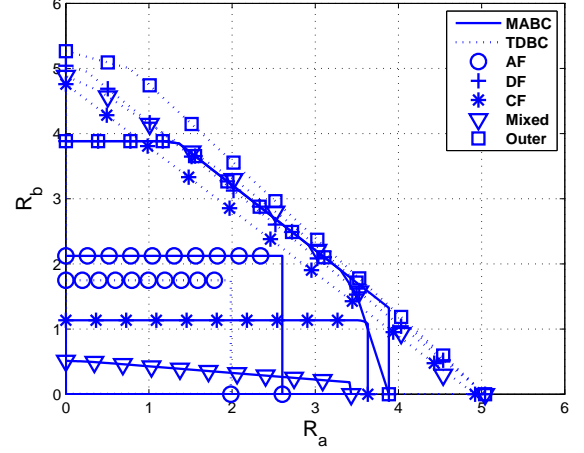


Fig. 11. Comparison of bi-directional regions with  $h_{ar} = 20$ ,  $h_{br} = 0.6$ ,  $h_{ab} = 0.5$ ,  $P_a = P_b = P_r = 20$  dB and  $N_a = N_b = N_r = 1$ .

for each link. In addition, the sum-rates in the constrained case where  $\sigma = 2$  (Fig. 17) are asymmetric even for non-mixed protocols. The intuitive reason for this is that the rates are constrained in an asymmetric way, and hence a particular, non-midpoint distance will be optimal even for CF and DF forwarding schemes. The performance of the  $\sigma = 2$  sum-rate for the DF MABC protocol is remarkably asymmetric where it peaks and almost touches the outer bound at  $\zeta = 0.6$ . These plots and region optimizations may be useful when determining the optimal relay position subject to particular rate constraints.

## VI. CONCLUSION

In this paper, we derived achievable rate regions for 4 new half-duplex bi-directional relaying protocols. We have specialized the 8 different achievable rate regions and the 2 outer bounds to the Gaussian case and numerically evaluated them under various channel conditions. For the MABC protocol, DF or CF is the optimal scheme, depending on the given channel and SNR regime. In the TDBC protocol, the relative performance of the forwarding schemes depends on the given channel condition. Notably, we have determined an example of a channel condition in which the mixed TDBC protocol outperforms the other proposed protocols. In general, the MABC protocol outperforms the TDBC protocol in the low SNR regime, while the reverse is true in the high SNR regime.

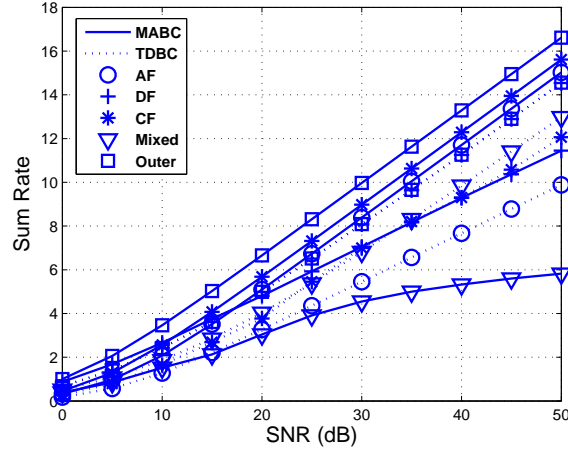


Fig. 12. Maximum sum-rate of the 8 bi-directional protocols and 2 outer bounds at different SNR. Here  $h_{ar} = h_{br} = 1$  and  $h_{ab} = 0.2$

## APPENDIX I

### PROOF OF THEOREM 14

*Proof: Random code generation:* For simplicity of exposition, we take  $|\mathcal{Q}| = 1$ .

1) Phase 1: Generate random  $(n \cdot \Delta_{1,n})$ -length sequences

- $\mathbf{x}_a^{(1)}(w_a)$  i.i.d. with  $p^{(1)}(x_a)$ ,  $w_a \in \mathcal{S}_a = \{0, 1, \dots, \lfloor 2^{nR_a} \rfloor - 1\}$

and generate a partition of  $\mathcal{S}_a$  randomly by independently assigning every index  $w_a \in \mathcal{S}_a$  to a set  $\mathcal{S}_{a,i}$ , with a uniform distribution over the indices  $i \in \{0, \dots, \lfloor 2^{nR_{a0}} \rfloor - 1\} := \mathcal{S}_{a0}$ . We denote by  $s_a(w_a)$  the index  $i$  of  $\mathcal{S}_{a,i}$  to which  $w_a$  belongs.

2) Phase 2: Generate random  $(n \cdot \Delta_{2,n})$ -length sequences

- $\mathbf{x}_b^{(2)}(w_b)$  i.i.d. with  $p^{(2)}(x_b)$ ,  $w_b \in \mathcal{S}_b = \{0, 1, \dots, \lfloor 2^{nR_b} \rfloor - 1\}$
- $\hat{\mathbf{y}}_r^{(2)}(w_{r0})$  i.i.d. with  $p^{(2)}(\hat{y}_r) = \sum_{y_r} p^{(2)}(y_r) p^{(2)}(\hat{y}_r | y_r)$ ,  $w_{r0} \in \{0, 1, \dots, \lfloor 2^{nR_{r0}} \rfloor - 1\} := \mathcal{S}_{r0}$

3) Phase 3: Generate random  $(n \cdot \Delta_{3,n})$ -length sequences

- $\mathbf{u}_r^{(3)}(w_r)$  i.i.d. with  $p^{(3)}(u_r)$ ,  $w_r \in \{0, 1, \dots, \lfloor 2^{nR_r} \rfloor - 1\} := \mathcal{S}_r$
- $\mathbf{u}_b^{(3)}(w_{r0})$  i.i.d. with  $p^{(3)}(u_b)$ ,  $w_{r0} \in \mathcal{S}_{r0}$

and define bin  $B_i := \{w_r | w_r \in [(i-1) \cdot \lfloor 2^{n(R_r - R_{a0})} \rfloor + 1, i \cdot \lfloor 2^{n(R_r - R_{a0})} \rfloor]\}$  for  $i \in \mathcal{S}_{a0}$ .

*Encoding:* During phase 1 (resp. phase 2), the encoder of node a (resp. b) sends the codeword  $\mathbf{x}_a^{(1)}(w_a)$  (resp.  $\mathbf{x}_b^{(2)}(w_b)$ ). At the end of phase 1, relay r estimates (or decodes)  $\tilde{w}_a$ . At the end of phase 2, the relay compresses the received  $\mathbf{y}_r^{(2)}$  to a message  $w_{r0}$  if there exists a  $w_{r0}$  such that  $(\mathbf{y}_r^{(2)}, \hat{\mathbf{y}}_r^{(2)}(w_{r0})) \in$

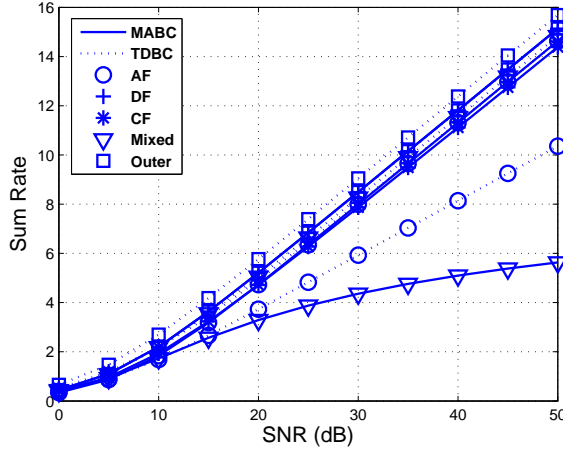


Fig. 13. Maximum sum-rate of the 8 bi-directional protocols and 2 outer bounds at different SNR. Here  $h_{ar} = 0.6$ ,  $h_{br} = 20$  and  $h_{ab} = 0.5$

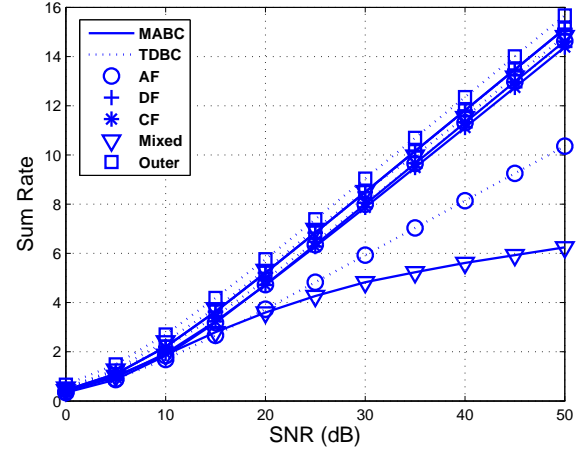


Fig. 14. Maximum sum-rate of the 8 bi-directional protocols and 2 outer bounds at different SNR. Here  $h_{ar} = 20$ ,  $h_{br} = 0.6$  and  $h_{ab} = 0.5$

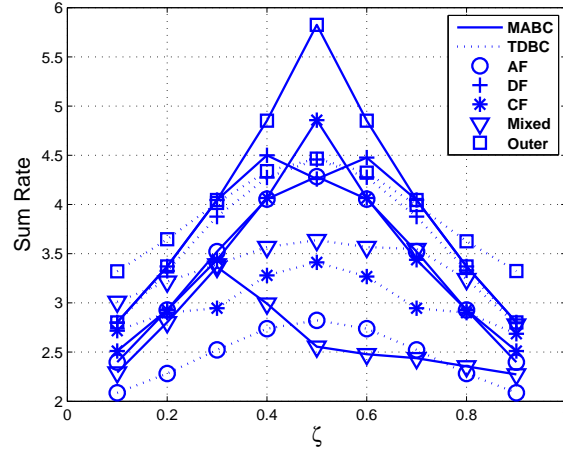


Fig. 15. Maximum sum-rate of the 8 bi-directional protocols at different relay position. Here  $h_{ab} = 0.2$  and no rate constraints.

$A^{(2)}(Y_r \hat{Y}_r)$ . Such an  $w_{r0}$  exists with high probability if

$$R_{r0} = \Delta_{2,n} I(Y_r^{(2)}; \hat{Y}_r^{(2)}) + \epsilon \quad (147)$$

and  $n$  is sufficiently large. Also we choose

$$R_r = \Delta_{3,n} I(U_r^{(3)}; Y_b^{(3)}) - 4\epsilon \quad (148)$$

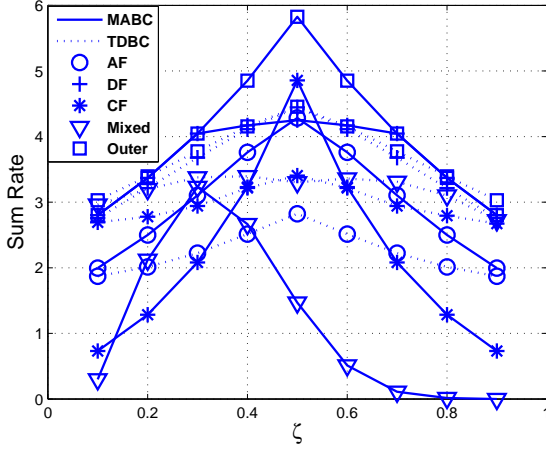


Fig. 16. Maximum sum-rate of the 8 bi-directional protocols at different relay position. Here  $h_{ab} = 0.2$  and  $\sigma = 1$  ( $R_a = R_b$ ).

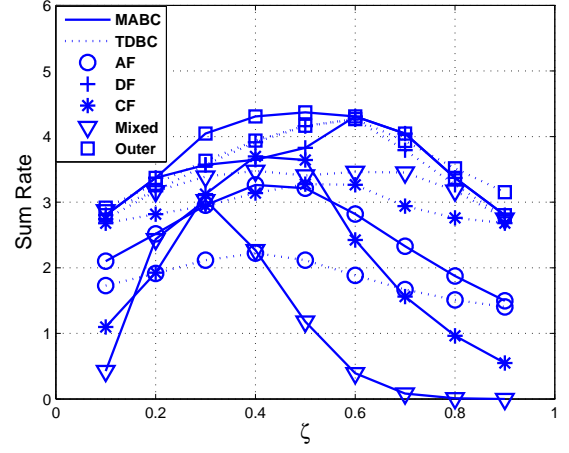


Fig. 17. Maximum sum-rate of the 8 bi-directional protocols at different relay position. Here  $h_{ab} = 0.2$  and  $\sigma = 2$  ( $R_a = 2R_b$ ).

To choose  $w_r$ , the relay first selects  $w_{a0} = s_a(\tilde{w}_a)$  and the bin  $B_{w_{a0}}$ , then it searches the minimum  $w_r \in B_{w_{a0}}$  such that  $(\mathbf{u}_r^{(3)}(w_r), \mathbf{u}_b^{(3)}(w_{r0})) \in A^{(3)}(U_r U_b)$ . This ensures uniqueness of  $w_r$  if such a  $w_r$  exists, i.e.,  $w_r$  is a function of  $(w_{a0}, w_{r0})$ . Such a  $w_r$  exists with high probability if

$$|B_{s_a(\tilde{w}_a)}| = 2^{n(\Delta_{3,n}I(U_r^{(3)}; U_b^{(3)}) + \epsilon)} \quad (149)$$

Since  $|B_j| = 2^{n(R_r - R_{a0})}$ ,  $\forall j \in \mathcal{S}_{a0}$ , the condition is equivalent to

$$R_{a0} < \Delta_{3,n}I(U_r^{(3)}; Y_b^{(3)}) - \Delta_{3,n}I(U_r^{(3)}; U_b^{(3)}) - 5\epsilon \quad (150)$$

The relay then sends  $\mathbf{x}_r^{(3)}$  randomly generated i.i.d. according to  $p^{(3)}(x_r | u_r, u_b)$  with  $\mathbf{u}_r^{(3)}(w_r)$  and  $\mathbf{u}_b^{(3)}(w_{r0})$  during phase 2.

*Decoding:* Node a estimates  $\tilde{w}_{r0}$  after phase 3 using jointly typical decoding. First, since node a knows  $w_a$ , it can reduce the cardinality of  $w_r$  to  $\lfloor 2^{n(R_r - R_{a0})} \rfloor$ . Furthermore, it forms two sets of  $\tilde{w}_{r0}$  based on typical sequences,  $\{\tilde{w}_{r0} | (\mathbf{y}_a^{(2)}, \hat{\mathbf{y}}_r^{(2)}(\tilde{w}_{r0})) \in A^{(2)}(Y_a \hat{Y}_r)\}$  and  $\{\tilde{w}_{r0} | (\mathbf{u}_r^{(3)}(\tilde{w}_r), \mathbf{u}_b^{(3)}(\tilde{w}_{r0}), \mathbf{y}_a^{(3)}) \in A^{(3)}(U_r U_b Y_a), \tilde{w}_r \in B_{w_{a0}}\}$ . After decoding  $\tilde{w}_{r0}$  (which is a success if there is a single common element in both of the previous sets), node a decodes  $\tilde{w}_b$  using jointly typical decoding of the sequence  $(\mathbf{x}_b^{(2)}, \hat{\mathbf{y}}_r^{(2)}, \mathbf{y}_a^{(2)})$ . Node b decodes  $\tilde{w}_r$  after phase 3 and from the bin index of  $\tilde{w}_r$  it estimates  $\tilde{s}_a$ . Then node b decodes the index as  $\tilde{w}_a$  if there exists a unique  $\tilde{w}_a \in \mathcal{S}_{a, \tilde{s}_a}$  such that  $(\mathbf{x}_a^{(1)}(\tilde{w}_a), \mathbf{y}_b^{(1)}) \in A^{(1)}(X_a Y_b)$ .

Error analysis:

$$P[E_{a,b}] \leq P[E_{a,r}^{(1)} \cup E_{r,b}^{(3)} \cup E_{a,b}^{(3)}] \quad (151)$$

$$\leq P[E_{a,r}^{(1)}] + P[E_{r,b}^{(3)} | \bar{E}_{a,r}^{(1)}] + P[E_{a,b}^{(3)} | \bar{E}_{a,r}^{(1)} \cap \bar{E}_{r,b}^{(3)}] \quad (152)$$

$$P[E_{b,a}] \leq P[E_{r,a}^{(3)} \cup E_{b,a}^{(3)}] \quad (153)$$

$$\leq P[E_{r,a}^{(3)}] + P[E_{b,a}^{(3)} | \bar{E}_{r,a}^{(3)}] \quad (154)$$

We define error events in each phase as follows:

- 1)  $E_{a,r}^{(1)} = E_{a,r}^{(1),1} \cup E_{a,r}^{(1),2}$ .  
 $E_{a,r}^{(1),1} : (\mathbf{x}_a^{(1)}(w_a), \mathbf{y}_r^{(1)}) \notin A^{(1)}(X_a Y_r)$ .  
 $E_{a,r}^{(1),2} : \text{there exists } \tilde{w}_a \neq w_a \text{ such that } (\mathbf{x}_a^{(1)}(\tilde{w}_a), \mathbf{y}_r^{(1)}) \in A^{(1)}(X_a Y_r)$ .
- 2)  $E_{r,b}^{(3)} = E_{r,b}^{(3),1} \cup E_{r,b}^{(3),2} \cup E_{r,b}^{(3),3} \cup E_{r,b}^{(3),4}$ .  
 $E_{r,b}^{(3),1} : \text{there does not exist a } w_{r0} \text{ such that } (\mathbf{y}_r^{(2)}, \hat{\mathbf{y}}_r^{(2)}(w_{r0})) \in A^{(2)}(Y_r \hat{Y}_r)$ .  
 $E_{r,b}^{(3),2} : \text{there does not exist a } w_r \in B_{w_{a0}} \text{ such that } (\mathbf{u}_r^{(3)}(w_r), \mathbf{u}_b^{(3)}(w_{r0})) \in A^{(3)}(U_r U_b)$ .  
 $E_{r,b}^{(3),3} : (\mathbf{u}_r^{(3)}(w_r), \mathbf{y}_b^{(3)}) \notin A^{(3)}(U_r Y_b)$ .  
 $E_{r,b}^{(3),4} : \text{there exists } \tilde{w}_r \neq w_r \text{ such that } (\mathbf{u}_r^{(3)}(\tilde{w}_r), \mathbf{y}_b^{(3)}) \in A^{(3)}(U_r Y_b)$ .
- 3)  $E_{a,b}^{(3)} = E_{a,b}^{(3),1} \cup E_{a,b}^{(3),2}$ .  
 $E_{a,b}^{(3),1} : (\mathbf{x}_a^{(1)}(w_a), \mathbf{y}_b^{(1)}) \notin A^{(1)}(X_a Y_b)$ .  
 $E_{a,b}^{(3),2} : \text{there exists } \tilde{w}_a \neq w_a \text{ such that } (\mathbf{x}_a^{(1)}(\tilde{w}_a), \mathbf{y}_b^{(1)}) \in A^{(1)}(X_a Y_b) \text{ and } \tilde{w}_a \in S_{a,w_{a0}}$ .
- 4)  $E_{r,a}^{(3)} = E_{r,a}^{(3),1} \cup E_{r,a}^{(3),2} \cup E_{r,a}^{(3),3} \cup E_{r,a}^{(3),4} \cup E_{r,a}^{(3),5} \cup E_{r,a}^{(3),6}$ .  
 $E_{r,a}^{(3),1} : \text{there does not exist a } w_{r0} \text{ such that } (\mathbf{y}_r^{(2)}, \hat{\mathbf{y}}_r^{(2)}(w_{r0})) \in A^{(2)}(Y_r \hat{Y}_r)$ .  
 $E_{r,a}^{(3),2} : \text{there does not exist a } w_r \in B_{w_{a0}} \text{ such that } (\mathbf{u}_r^{(3)}(w_r), \mathbf{u}_b^{(3)}(w_{r0})) \in A^{(3)}(U_r U_b)$ .  
 $E_{r,a}^{(3),3} : (\mathbf{y}_a^{(2)}, \hat{\mathbf{y}}_r^{(2)}(w_{r0})) \notin A^{(2)}(Y_a \hat{Y}_r)$ .  
 $E_{r,a}^{(3),4} : (\mathbf{u}_r^{(3)}(w_r), \mathbf{u}_b^{(3)}(w_{r0}), \mathbf{y}_a^{(3)}) \notin A^{(3)}(U_r U_b Y_a)$ .  
 $E_{r,a}^{(3),5} : \text{there exists } (\tilde{w}_r, \tilde{w}_{r0}) \text{ where } \tilde{w}_r \neq w_r \text{ and } \tilde{w}_{r0} \neq w_{r0} \text{ such that } (\mathbf{u}_r^{(3)}(\tilde{w}_r), \mathbf{u}_b^{(3)}(\tilde{w}_{r0}), \mathbf{y}_a^{(3)}) \in A^{(3)}(U_r U_b Y_a) \text{ and } (\mathbf{y}_a^{(2)}, \hat{\mathbf{y}}_r^{(2)}(\tilde{w}_{r0})) \in A^{(2)}(Y_a \hat{Y}_r)$ . Recall,  $w_r$  is uniquely specified by  $(w_{a0}, w_{r0})$ . Hence for a given  $w_{a0}$ , there are at most  $2^{nR_{r0}}$  such  $(\tilde{w}_r, \tilde{w}_{r0})$  pairs.  
 $E_{r,a}^{(3),6} : \text{there exists } \tilde{w}_{r0} \neq w_{r0} \text{ such that } (\mathbf{u}_r^{(3)}(w_r), \mathbf{u}_b^{(3)}(\tilde{w}_{r0}), \mathbf{y}_a^{(3)}) \in A^{(3)}(U_r U_b Y_a),$   
 $(\mathbf{y}_a^{(2)}, \hat{\mathbf{y}}_r^{(2)}(\tilde{w}_{r0})) \in A^{(2)}(Y_a \hat{Y}_r)$ .
- 5)  $E_{b,a}^{(3)} = E_{b,a}^{(3),1} \cup E_{b,a}^{(3),2}$ .  
 $E_{b,a}^{(3),1} : (\mathbf{x}_b^{(2)}(w_b), \mathbf{y}_a^{(2)}, \hat{\mathbf{y}}_r^{(2)}(w_{r0})) \notin A^{(2)}(X_b Y_a \hat{Y}_r)$ .

$E_{b,a}^{(3),2}$  : there exists  $\tilde{w}_b \neq w_b$  such that  $(\mathbf{x}_b^{(2)}(\tilde{w}_b), \mathbf{y}_a^{(2)}, \hat{\mathbf{y}}_r^{(2)}(w_{r0})) \in A^{(2)}(X_b Y_a \hat{Y}_r)$ .

Then,

$$P[E_{a,r}^{(1)}] \leq P[E_{a,r}^{(1),1}] + P[E_{a,r}^{(1),2}] \quad (155)$$

$$\leq P[\bar{D}^{(1)}(\mathbf{x}_a(w_a), \mathbf{y}_r)] + P[\cup_{\tilde{w}_a \neq w_a} D^{(1)}(\mathbf{x}_a(\tilde{w}_a), \mathbf{y}_r)] \quad (156)$$

$$\leq \epsilon + 2^{n(R_a - \Delta_{1,n} I(X_a^{(1)}; Y_r^{(1)}) + 3\epsilon)} \quad (157)$$

$$P[E_{r,b}^{(3)} | \bar{E}_{a,r}^{(1)}] \leq P[E_{r,b}^{(3),1} | \bar{E}_{a,r}^{(1)}] + P[E_{r,b}^{(3),2} | \bar{E}_{a,r}^{(1)}] + P[E_{r,b}^{(3),3} | \bar{E}_{a,r}^{(1)}] + P[E_{r,b}^{(3),4} | \bar{E}_{a,r}^{(1)}] \quad (158)$$

$$\leq 2\epsilon + P[\bar{D}^{(3)}(\mathbf{u}_r(w_r), \mathbf{y}_b)] + P[\cup_{\tilde{w}_r \neq w_r} D^{(3)}(\mathbf{u}_r(\tilde{w}_r), \mathbf{y}_b)] \quad (159)$$

$$\leq 3\epsilon + 2^{n(R_r - \Delta_{3,n} I(U_r^{(3)}; Y_b^{(3)}) + 3\epsilon)} \quad (160)$$

In (159),  $P[E_{r,b}^{(3),1} | \bar{E}_{a,r}^{(1)}]$  and  $P[E_{r,b}^{(3),2} | \bar{E}_{a,r}^{(1)}]$  are less than  $\epsilon$  due to (147) and (150), respectively.

$$P[E_{a,b}^{(3)} | \bar{E}_{a,r}^{(1)} \cap \bar{E}_{r,b}^{(3)}] \leq P[E_{a,b}^{(3),1} | \bar{E}_{a,r}^{(1)} \cap \bar{E}_{r,b}^{(3)}] + P[E_{a,b}^{(3),2} | \bar{E}_{a,r}^{(1)} \cap \bar{E}_{r,b}^{(3)}] \quad (161)$$

$$\leq P[\bar{D}^{(1)}(\mathbf{x}_a(w_a), \mathbf{y}_b)] + P[\cup_{\tilde{w}_a \neq w_a} D^{(1)}(\mathbf{x}_a(w_a), \mathbf{y}_b), s_a(w_a) = s_a(\tilde{w}_a)] \quad (162)$$

$$\leq \epsilon + 2^{n(R_a - \Delta_{1,n} I(X_a^{(1)}; Y_b^{(1)}) - R_{a0} + 3\epsilon)} \quad (163)$$

$$P[E_{r,a}^{(3)}] \leq P[E_{r,a}^{(3),1}] + P[E_{r,a}^{(3),2}] + P[E_{r,a}^{(3),3}] + P[E_{r,a}^{(3),4}] + P[E_{r,a}^{(3),5}] + P[E_{r,a}^{(3),6}] \quad (164)$$

$$\leq 2\epsilon + P[\bar{D}^{(2)}(\mathbf{y}_a, \hat{\mathbf{y}}_r(w_{r0}))] + P[\bar{D}^{(3)}(\mathbf{u}_r(w_r), \mathbf{u}_b(w_{r0}), \mathbf{y}_a)] +$$

$$P[\cup_{\substack{\tilde{w}_r \neq w_r \\ \tilde{w}_{r0} \neq w_{r0}}} D^{(3)}(\mathbf{u}_r(\tilde{w}_r), \mathbf{u}_b(\tilde{w}_{r0}), \mathbf{y}_a), D^{(2)}(\mathbf{y}_a, \hat{\mathbf{y}}_r(\tilde{w}_{r0}))] +$$

$$P[\cup_{\substack{\tilde{w}_r = w_r \\ \tilde{w}_{r0} \neq w_{r0}}} D^{(3)}(\mathbf{u}_r(w_r), \mathbf{u}_b(\tilde{w}_{r0}), \mathbf{y}_a), D^{(2)}(\mathbf{y}_a, \hat{\mathbf{y}}_r(\tilde{w}_{r0}))] \quad (165)$$

$$\leq 3\epsilon + 2^{n(R_{r0} - \Delta_{3,n} I(U_r^{(3)}, U_b^{(3)}; Y_a^{(3)}) - \Delta_{2,n} I(Y_a^{(2)}; \hat{Y}_r^{(2)}) + 7\epsilon) +}$$

$$2^{n(R_{r0} - \Delta_{3,n} I(U_b^{(3)}; U_r^{(3)}, Y_a^{(3)}) - \Delta_{2,n} I(Y_a^{(2)}; \hat{Y}_r^{(2)}) + 7\epsilon)} \quad (166)$$

In (165),  $P[E_{r,a}^{(3),1}]$  and  $P[E_{r,a}^{(3),2}]$  are less than  $\epsilon$  due to (147) and (150), respectively. In (166),  $P[\bar{D}^{(2)}(\mathbf{y}_a, \hat{\mathbf{y}}_r(w_{r0}))]$  is less than  $\epsilon$  by the Markov lemma.

$$P[E_{b,a}^{(3)} | \bar{E}_{r,a}^{(3)}] \leq P[E_{b,a}^{(3),1} | \bar{E}_{r,a}^{(3)}] + P[E_{b,a}^{(3),2} | \bar{E}_{r,a}^{(3)}] \quad (167)$$

$$\leq P[\bar{D}^{(2)}(\mathbf{x}_b(w_b), \mathbf{y}_a, \hat{\mathbf{y}}_r(w_{r0}))] + P[\cup_{\tilde{w}_b \neq w_b} D^{(2)}(\mathbf{x}_b(\tilde{w}_b), \mathbf{y}_a, \hat{\mathbf{y}}_r(w_{r0}))] \quad (168)$$

$$\leq \epsilon + 2^{n(R_b - \Delta_{2,n} I(X_b^{(2)}; Y_a^{(2)}, \hat{Y}_r^{(2)}) + 4\epsilon)} \quad (169)$$

Since  $\epsilon > 0$  is arbitrary, the proper choice of  $R_{a0}$ , the conditions of Theorem 14, (148), (150) and the AEP guarantee that the right hand sides of (157), (160), (163), (166) and (169) corresponding to the



first term of (84), (148), (the second term of (84) and (150)), (86) and (85) vanish as  $n \rightarrow \infty$ . By the Carathéodory theorem in [9], it is sufficient to restrict  $|Q| \leq 6$ . ■

## REFERENCES

- [1] R. Ahlswede, N. Cai, S.-Y. R. Li, and R. W. Yeung, "Network information flow," *IEEE Trans. Inform. Theory*, vol. 46, pp. 1204–1216, 2000.
- [2] T. Berger, *Multiterminal source coding*. New York: Springer Verlag, 1977.
- [3] M. Costa, "Writing on dirty paper," *IEEE Trans. Inform. Theory*, vol. IT-29, pp. 439–441, May 1983.
- [4] T. Cover and J. Thomas, *Elements of Information Theory*, 2nd ed. New York: Wiley, 2006.
- [5] T. Cover and A. E. Gamal, "Capacity theorems for the relay channel," *IEEE Trans. Inform. Theory*, vol. IT-25, pp. 572–584, Sept. 1979.
- [6] A. E. Gamal and E. van der Meulen, "A proof of marton's coding theorem for the discrete memoryless broadcast channel," *IEEE Trans. Inform. Theory*, vol. IT-27, pp. 120–122, Jan. 1981.
- [7] S. Gel'fand and M. Pinsker, "Coding for channels with random parameters," vol. 9, no. 9, pp. 19–31, Jan. 1980.
- [8] D. Gunduz, E. Tuncel, and J. Nayak, "Rate regions for the seperated two-way relay channel," in *46th Annual Allerton Conference on Communication, Control, and Computing*, Monticello, IL, Sept. 2008.
- [9] J. B. Hiriart-Urruty and C. Lemaréchal, *Fundamentals of Convex Analysis*. New York; Springer, 2001.
- [10] S. J. Kim, P. Mitran, and V. Tarokh, "Performance bounds for bi-directional coded cooperation protocols," *IEEE Trans. Inform. Theory*, vol. 54, no. 11, pp. 5235–5241, Nov. 2008.
- [11] G. Kramer, M. Gastpar, and P. Gupta, "Cooperative strategies and capacity theorems for relay networks," *IEEE Trans. Inform. Theory*, vol. 51, no. 9, pp. 3037–3063, Sep. 2005.
- [12] P. Larsson, N. Johansson, and K.-E. Sunell, "Coded bi-directional relaying," in *Proc. IEEE Veh. Technol. Conf. - Spring*, 2006, pp. 851–855.
- [13] K. Marton, "A coding theorem for the discrete memoryless broadcast channel," *IEEE Trans. Inform. Theory*, vol. 25, pp. 306–311, May 1979.
- [14] K. Narayanan, M. P. Wilson, and A. Sprintson, "Joint physical layer coding and network coding for bi-directional relaying," in *45th Annual Allerton Conference*, Monticello, IL, Sept. 2007.
- [15] J. Nayak, E. Tuncel, and D. Gunduz, "Wyner-ziv coding over broadcast channels," in *IEEE Inform. Theory Workshop*, Porto, Portugal, 2008.
- [16] T. J. Oechtering, C. Schnurr, I. Bjelakovic, and H. Boche, "Achievable rate region of a two phase bidirectional relay channel," in *Proc. Conf. on Inf. Sci. and Sys.*, Baltimore, MD, Mar. 2007.
- [17] Y. Oohama, "Gaussian multiterminal source coding," *IEEE Trans. Inform. Theory*, vol. 43, no. 6, pp. 1912–1923, Nov. 1997.
- [18] P. Popovski and H. Yomo, "The anti-packets can increase the achievable throughput of a wireless multi-hop network," in *Proc. IEEE Int. Conf. Commun.*, 2006, pp. 3885–3890.
- [19] —, "Bi-directional amplification of throughput in a wireless multi-hop network," in *Proc. IEEE Veh. Technol. Conf. - Spring*, 2006, pp. 588–593.
- [20] B. Rankov and A. Wittneben, "Spectral efficient signaling for half-duplex relay channels," in *39th Asilomar Conference on Signals, Systems, and Computers*, Pacific Grove, CA, Nov. 2005.

- [21] C. Schnurr, T. J. Oechtering, and S. Stanczak, "Achievable rates for the restricted half-duplex two-way relay channel," in *41st Asilomar Conference on Signals, Systems and Computers*, Monterey, CA, Nov. 2007.
- [22] C. E. Shannon, "Two-way communications channels," in *4th Berkeley Symp. Math. Stat. Prob.*, Chicago, IL, June 1961, pp. 611–644.
- [23] E. Tuncel, "Slepian-wolf coding over broadcast channels," *IEEE Trans. Inform. Theory*, vol. 52, no. 4, pp. 1469–1482, April 2006.